AN

ESSAY

ON

MATHEMATICAL LANGUAGE;

OR, AN

INTRODUCTION

TO THE

MATHEMATICAL SCIENCES.

BY

C. G. A. BASELLI.

LONDON:

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PREFACE.

THE Author of this Work confiders the different Branches of the Mathematics as so many Languages; and they are so in effect, for men have invented languages to express their thoughts by outward signs, and thereby to mark the objects of ideas, their properties and relations. In like manner, Mathematicians have adopted signs to mark different quantities, and to express their properties and relations.

Now, since these signs are either *Numbers*, *Letters*, or *Lines*, there are consequently three primitive Mathematical Languages; namely, *Arithmetic*, in which numbers; *Algebra*, wherein letters; and *Geometry*, in which lines are used. But since we may, at the same time, make use of numbers and letters, of letters and lines, there are two other languages, which being derivatives,

may

may be called *Arithmetico-Algebraic*, and *Algebraico-Geometric:* and ſince the eye of the immortal Newton has penetrated to the firſt elements of quantity, which he calls *Fluxions*, there is a ſixth Language, which though ſtrictly ſpeaking it be *Algebraico-Geometric*, may, however, take a more particular name from its object, and therefore be called *Fluxional*, or *Tranſcendental*.

In each of theſe Languages, the Author explains the manner of *reading*, *writing*, and *ſpeaking* mathematically, and thus brings the ſcience within the reach of every common genius.

We *read mathematically*, when we expreſs in words what is repreſented by mathematical ſigns: We *write mathematically*, when we repreſent by ſigns what has been expreſſed only in words: and, laſtly, we *ſpeak mathematically*, when we make uſe of ſigns in the inveſtigation and demonſtration of theorems, and in the ſolution of problems; thereby fixing the principles and rules for learning this ſcience without any aſſiſtance, and carrying it to the higheſt degree of perfection.

The experience of former years has convinced the Author, that this method of ſtudying the Mathematics, provided the pupil will give a proper degree of attention, and have a little patience at the beginning, ſerves not only to make him acquire

quire the ſcience very eaſily, but alſo to develope his genius, rectify his judgement, give him a taſte for invention, and actually to make him become inſenſibly an Inventor.

This is the reaſon which has determined the Author to offer his method to the Public, and to treat the ſublime Science of Mathematics as a Language, with a view, not of giving a compleat ſyſtem, but of explaining the principles, in order to lay ſuch a foundation, as that a beginner may, without any other aſſiſtance, make a competent progreſs in the Mathematics. Nor is it a vain preſumption which has led the Author to this determination, but a conviction founded on his own experience in Italy, Switzerland, and England; where, by giving lectures after this method, he has always had the ſatisfaction in a few weeks to make his pupils fond of the Mathematics, and capable of purſuing their ſtudies by themſelves. Add to this, the judgement of two celebrated Mathematicians of the Univerſity of Cambridge, who having been ſo obliging as to read the firſt part of this Work, have given it as their opinion, *that it is ingenious, that the ſubject is treated with more than uſual accuracy, and that it will be uſeful to the Public.*

It may not be amiſs alſo to obſerve, that this Work contains ſeveral things which are new: as, 1ſt, Rules for ſimplifying the common operations in Arithmetic; 2dly, Several general rules for reſolving all problems, of whatever degree, that can be propoſed in numbers, by Arithmetic only, without Algebra; 3rdly, A very eaſy method of reſolving a numerical equation of any degree, in all caſes at leaſt wherein it has a *real* and commenſurable root; 4thly, A new manner of determining *incommenſurable* quadratic roots; 5thly, A method of approximating to the value of an incommenſurable but real root; 6thly, A general rule for making the expreſſion $\sqrt{(a+bx+cx^2)}$ rational in numbers; 7thly, and laſtly, A direct and very ſimple ſolution of this problem—*the general term of an algebraic ſeries being given, to find an expreſſion for the ſum of the ſeries.*

The method of converting any given ſimple equation into three other ſimple ones, leads to a capital diſcovery in Mathematics, which is, that *every ſimple equation muſt have four different roots, two real, and two imaginary.*

The Author has an intention of publiſhing a particular Diſſertation upon his Diſcovery, wherein he will demonſtrate its uſe in mathematical Problems, and chiefly in the conſtruction of

curve

curve Lines. Now he obſerves only, that the *Law of Continuation* is rigorouſly obſerved by theſe Lines, ſo that, for inſtance, the two Branches of the common Hyperbola are joined together by a *nodus*.

The firſt part only, containing the principles of Arithmetic and Algebra, and being an entire work of itſelf, is now offered to the Public. The ſecond, explaining the methods of treating elementary and ſublime Geometry, and Fluxions, will follow, if the preſent volume ſhould meet with general approbation.

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INTRODUCTION.

Nature and Division of the Mathematical Language.

THE Mathematical Language consists of the knowledge and use of some characters and signs, invented to represent the different quantities, which are the object of Mathematics, to denote the operations to be performed with them, and to express their properties and the relations arising from their comparison. And because the proper objects of the Mathematical science are *number*, *extension*, and *quantity in general*; three kinds of characters are used in exhibiting these different objects, viz. the Arabic figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, to denote the numbers, lines to represent the extension, and the letters of the alphabet to express quantities in general.

Again, in order to mark the operations to be performed upon magnitude, and to express its relations, the following signs are taken.

$+$ *Plus* or *more*. The sign of Addition. Thus $3+5$ denotes the sum of 3 and 5, or 8. When no sign is expressed, $+$ is understood. Thus 8, and $+8$ mean the same thing.

− *Minus* or *less*. The sign of Subtraction. Thus $6-2$ denotes the excess of 6 above 2, that is 4.

Note. The quantities, connected by these characters, are said to be *compound*, as $3+5$, and $6-2$.

× *Multiplied by*. The sign of Multiplication. Thus 4×6 denotes the product of the numbers 4 and 6, that is 24.

The multiplication of quantities is also expressed by setting a point between them, and 4.6 is therefore the same thing as 4×6.

Note. The products arising from the continual multiplication of the same quantity, are called the *powers* of that quantity, which is the *root*. Thus 2.2, 2.2.2, 2.2.2.2, &c. are *powers* of the *root* 2.

These powers are expressed by placing above the root, to the right hand, a smaller figure, denoting how often the root is repeated. This figure is called an *index* or *exponent*, and from it the power is denominated. Thus,

$$\left.\begin{array}{r}2\\2.2\\2.2.2\\2.2.2.2\end{array}\right\}\ \text{is called the}\ \left\{\begin{array}{l}\text{1st}\\\text{2d}\\\text{3d}\\\text{4th}\end{array}\right\}\ \begin{array}{l}\textit{power}\ \text{of the}\\\textit{root}\ 2,\ \text{and is}\\\text{otherwise ex-}\\\text{pressed by}\end{array}\ \left\{\begin{array}{l}2^1\ \text{or}\ 2\\2^2\\2^3\\2^4\end{array}\right.$$

The 2d and 3d powers are generally called the *square* and *cube*; and the 4th, 5th, and 6th, are also sometimes respectively called the *biquadrate*, *sursolid*, and *cubocube*.

÷ *Divided by*. The sign of Division. Thus $8\div 2$ denotes the *quotient* of the former of the numbers divided by the latter, that is 4.

Division is also marked thus 2)8, the latter number being the *dividend*, and the former the *divisor*.

The *quotient* of two quantities is also denoted by placing the *dividend* above a small line, and the *divisor* below it. Thus $\frac{8}{2}$ is the quotient of 8 divided by 2, or 4. This expression of a quotient is also called a *fraction*.

√ This

$\surd$ This mark is called the *radical ſign*, and has a figure or *index* ſet over it, to denote what *root* it is of the number before which it ſtands. Thus, $\sqrt[2]{4}$ or $\surd 4$ ſignifies the 2d or ſquare root of 4; $\sqrt[3]{8}$ denotes the 3d or cube root of 8.

A *vinculum* is a line drawn over any number of terms of a compound quantity, to denote thoſe which are underſtood to be affected by the particular ſign connected with it. Thus, $\overline{8-2}\times 3$ ſhews that the exceſs of 8 above 2 muſt be multiplied by 3. Without the vinculum, the expreſſion $8-2\times 3$, would mean the exceſs of 8 above the product of 2 by 3. Therefore, the value of $\overline{8-2}\times 3$ is 18, and the value of $8-2\times 3$ is 2.

By ſome authors a *parentheſis* ( ) is uſed as a vinculum, and $(8-2)\times 3$ is the ſame thing as $\overline{8-2}\times 3$.

$=$ *Equal.* The ſign of equality. Thus $3+5=10-2$ means that the ſum of 3 and 5 is equal to the exceſs of 10 above 2.

$>$ *Greater*. The ſign of majority. Thus $3+5>8-2$ means that the ſum of 3 and 5 is greater than the exceſs of 8 above 2.

$<$ *Leſs*. The ſign of minority. Thus $3+5<12-2$ means that the ſum of 3 and 5 is leſs than the exceſs of 12 above 2.

: *To*, and : : *ſo is*. The ſigns of proportion. Thus $2:4::3:6$; that is, as 2 is to 4, ſo is 3 to 6.

Laſtly, it is by means of theſe different characters and ſigns, that the properties of magnitude are mathematically expreſſed. Thus $8=2\times 4$ ſhews that 8 is either double of 4, or quadruple of 2.

From the nature and combination of the characters uſed for expreſſing the quantities, it follows that the mathematical language may be divided into ſix others. Thus, when the Arabic figures are employed, the language is *arithmetical*; when lines, *geometrical*; when

the alphabetical letters, *algebraical*; when figures and lines, *arithmetico-geometrical*; when figures and letters, *arithmetico-algebraical*; and, laſtly, when letters and lines, *algebraico-geometrical*.

Theſe languages ariſe naturally from the mathematical expreſſions of finite magnitude. But there are exceedingly ſmall quantities, whoſe expreſſion, being quite different from the preceding ones, requires a particular alphabet and language, which I call *tranſcendental*, and will explain in its proper place.

The ſigns which ſerve to mark the operations upon quantities, and to expreſs the relations of magnitude, no more belong to one language than to another, being common to all; and therefore they ſhould be called *mathematical ſigns*.

From theſe principles it appears, that nobody can be a mathematician without a perfect knowledge of theſe languages; and no one, who would make any progreſs in the ſcience of magnitude, ought to be ignorant of their elements, and how to put in practice at leaſt their firſt rules. It is for this reaſon I ſhall try to explain the nature and principles of theſe languages, in order to facilitate and render more agreeable to my pupil a ſtudy, which, without theſe helps, has not always charms enough to attach a beginner to it. And ſince you are not thought to underſtand a language when you do not know its alphabet, and the way to write, read, and ſpeak it, I ſhall ſtop to explain minutely theſe articles in every one of our languages; and then I ſhall apply them to ſome of the moſt uſeful and moſt common examples, ſo that there may be gathered from the whole a ſufficient courſe of elementary mathematical leſſons.

PART

PART I.

THE ARITHMETICAL LANGUAGE.

CHAP. I.

The Nature of this Language.

I. *Arithmetical Alphabet.*

FROM the ten fingers of the hands, on which it had been usual to compute numbers, the figures were called *digits*. Their form, order, and value, are as follow:

1 one, an unit, or unity, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine, 0 cypher, nought, null, or nothing. Of these, the first nine, in contradistinction to the cypher, are called *significant figures.*

The value of the figures now assigned is called their *simple value*, as being that which they have in themselves, or when they stand alone. But when two or more figures are joined as in a line, the figures then receive also a local value from the place in which they stand, reckoning

koning the order of places from the right hand towards the left; thus,

&c.	Twelfth place.	Eleventh place.	Tenth place.	Ninth place.	Eighth place.	Seventh place.	Sixth place.	Fifth place.	Fourth place.	Third place.	Second place.	Firſt place.
	7	7	7	7	7	7	7	7	7	7	7	7

A figure ſtanding in the firſt place has only its ſimple value; but a figure in the ſecond place has ten times the value it would have in the firſt place, and a figure in the third place has ten times the value it would have in the ſecond place; and univerſally a figure in any ſuperior place has ten times the value it would have in the next inferior place.

Hence it is plain, that a figure in the firſt ſimply ſignifies ſo many units as the figure expreſſes; but the ſame figure advanced to the ſecond place, will ſignify ſo many tens; in the third place, it will ſignify ſo many hundreds; in the fourth place, ſo many thouſands; in the fifth place, ſo many ten thouſands; in the ſixth place, ſo many hundred thouſands; and in the ſeventh place, ſo many millions, &c. Thus, 7 in the firſt place will denote ſeven units; in the ſecond place, ſeven tens, or ſeventy; in the third place, ſeven hundred; in the fourth place, ſeven thouſand, &c.

Every three places, reckoning from the right hand, make a half-period; and the right-hand figures of theſe half-periods are termed *units* and *thouſands* by turns; the middle figure is always tens, and the left-hand figure always hundreds. But here obſerve, that the right-hand figure of every half-period, properly and ſtrictly ſpeaking, is always units; for the place called

thousands, when expressed at more length, is termed *units of thousands*, or the *unit's place of thousands*.

Two half-periods, or six places, make a full period; and the periods, reckoning from the right hand towards the left, are titled as follows, viz. the first is the period of *units*; the second, that of *millions*; the third is titled *bimillions* or *billions*; the fourth, *trimillions* or *trillions*; the fifth, *quadrillions*; the sixth, *quintillions*; the seventh, *sextillions*; the eighth, *septillions*; the ninth, *octillions*; the tenth, *nonillions*, &c.

Half-periods are usually distinguished from one another by a comma, and full periods by a point or colon; as in the table following.

TABLE.

4th period.						3d period.						2d period.						1st period.					
Trillions.						Billions.						Millions.						Units.					
Hundred Thousands.	Ten Thousands.	Thousands.	Hundreds.	Tens.	Units.	Hundred Thousands.	Ten Thousands.	Thousands.	Hundreds.	Tens.	Units.	Hundred Thousands.	Ten Thousands.	Thousands.	Hundreds.	Tens.	Units.	Hundred Thousands.	Ten Thousands.	Thousands.	Hundreds.	Tens.	Units.
9	6	4,	0	8	5.	8	1	3,	7	0	0.	2	3	7,	8	9	4.	6	7	8,	0	4	0

The table may be expressed in a more concise form thus,

4.	3.	2.	1. per.
Trillions.	Billions.	Millions.	Units.
$\bar{C}\bar{X}$M, CXU :	$\bar{C}\bar{X}$M, CXU :	$\bar{C}\bar{X}$M, CXU :	$\bar{C}\bar{X}$M, CXU.
964,085 :	813,700 :	237,894 :	678,040.

From the table it is obvious, that though a cypher signify nothing of itself, yet it serves to supply vacant places,

places, and raises the value of significant figures on its left hand, by throwing them into higher places. Thus, in the first period, by a cypher's filling the place of units, the figure 4 is thrown into the place of tens, and signifies forty. And a cypher likewise supplying the place of hundreds, the figure 8 is advanced to the next higher place, and signifies eight thousand. Again, two cyphers being in the places of tens and units of billions, the figure of 7 belongs to the next higher place, and signifies seven hundred billions, &c. But a cypher does not change the value of a significant figure on its right hand. Thus, 07 or 007 is the same as 7.

From this alphabet it is plain, that the value of figures increase from the right to the left, and decrease from the left to the right, in a decuple proportion, so that, by carrying on the places from that of units towards the right hand, the figures belonging to these places become ten times less for every place they are removed to the right, and consequently they express parts of unity, which may be properly called *decimal fractions*, or only *decimals*. Therefore, reckoning the order of these new places from the left hand towards the right, a figure standing in the first place will signify so many *tenth parts*, or *tenths*; but the same figure advanced to the second place, becomes so many *hundredth parts* or *hundredths*, and being removed one place more, it becomes *thousandth parts* or *thousandths*, &c. On the contrary, any decimal figure, by being removed one place toward the left, becomes ten times greater.

Hence equidistant places on the left and right of the place of units come under similar names, viz. tens and tenths, hundreds and hundredths, thousands and thousandths, &c. as in the following table.

Units

&c.	C Millions.	X Millions.	Millions.	C Thousands.	X Thousands.	Thousands.	Hundreds.	Tens.	Units.	Tenths.	Hundredths.	Thousandths.	X Thousandths.	C Thousandths.	Millionths.	X Millionths.	C Millionths.	&c.

It is usual to prefix a comma or point to Decimals, in order to distinguish them from a whole number. So .7 is seven tenths, .07 seven hundredths, .25 twenty-five hundredths, 3.75 three units and seventy-five hundredths, or three hundred seventy-five hundredths, &c. The point thus prefixed is called the *decimal point*.

In Decimals the figure next the point, being the first decimal place, is sometimes called *primes*, and the second figure from the point is called *seconds*, the next *thirds*, &c. Thus, in .875 the figure 8 is primes, 7 is seconds, and 5 is thirds.

An integer, by annexing cyphers, is raised to higher places on the left, and may, by this means, have its value increased to infinity. On the other hand, a decimal, by prefixing cyphers, is depressed to lower places on the right, and may, by this means, have its value diminished to infinity.

Cyphers annexed to decimals do not change the value of the decimals. Thus, .5 = .50 = .500 = .5000 = &c. mean always the same value, or five tenths.

II. *Descriptions of the Kinds and Species of Numbers.*

THE species of numbers are very various and manifold; but in this place it will be sufficient to describe such as appear most useful and necessary, particularly those that will occur in the ensuing treatise.

1. An *integer*, or *whole number*, is an unit, or any multitude of units; as 1, 7, 48, 100.

2. A *fraction*, or *broken number*, is any part or parts of an unit; and is expressed by two numbers, which are separated from one another by a line drawn betwixt them; the under number being called the *denominator*, and the upper one the *numerator*, of the fraction; as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{9}{10}$, $\frac{5}{100}$. The denominator gives name to the fraction, because it shews into how many parts the unit is divided; and the numerator tells how many of these parts the fraction contains. The numerator and denominator are in general named the *terms* of the fraction.

Note. A decimal is a fraction whose denominator is 1 with some number of cyphers annexed. Thus $\frac{9}{10}$, $\frac{5}{100}$, are decimal fractions. Therefore, $\frac{9}{10}$ and .9, $\frac{5}{100}$ and .05, are one and the same thing.

3. A *mixt number* is an integer with a fraction joined to it; as $4\frac{1}{3}$, $7\frac{3}{4}$, $4\frac{7}{10}$ or 4.7.

4. A number is said to *measure* another number, when it is contained in that other number a certain number of times, or when it divides that other number without any remainder. Thus 3 measures 6, 9, or 12.

5. An *even number* is that which is measured by 2, or which 2 divides without any remainder; as 2, 4, 6, 8, 10, 12.

6. An *odd number* is that which 2 does not measure, or which cannot be divided by 2 without a remainder; as 1, 3, 5, 7, 9, 11, 13.

7. A *prime number* is that which unity, or itself, only measures. Here is a table of prime numbers under 500.

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199.

211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293.
307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397.
401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499.

8. A *compoſite number* is that which is meaſured by ſome other number than itſelf, or unity; as 12, which is meaſured by 2, 3, 4, or 6.

9. Numbers are called *prime* to one another, when unity only meaſures them. Thus 13 and 31 are prime to one another; for no number, except unity, meaſures both.

10. Numbers are called *compoſite* to one another, when ſome number, beſides unity, meaſures them. Thus 12 and 18 are compoſite to one another, for 3 and 6 meaſures both of them.

11. A number which meaſures another is called an *aliquot part* of the other. Thus 6 is an aliquot part of 18, and 3 of 12, and 5 of 20.

12. The number meaſured, or which contains the aliquot part a certain number of times, is called a *multiple* of that aliquot part. Thus 18 is a multiple of 6, and 12 of 3, and 20 of 5.

13. A number is called an *aliquant part* of another, when it does not divide that other without a remainder. Thus 7 is an aliquant part of 24.

14. Two, three, or more numbers, which multiplied together produce another number, are called the *component parts* of the number produced. Thus 3 and 4, or 2 and 6, are the component parts of 12; and 2, 3, and 4, are the component parts of 24.

15. The product of a number multiplied into itſelf is called the *ſquare* or 2*d power* of that number; and the number itſelf is in this caſe called the *root:* and if the

 ſquare

ſquare be multiplied into the root, the product is called the *cube* or 3*d power* of that number: and if the cube be multiplied into the root, the product thence ariſing is called the *biquadrate* or 4*th power*, &c. The ſame thing has been put in another point of view before, when the manner in which the different powers are expreſſed, was explained.

CHAP. II.

OF WRITING ARITHMETICALLY.

DEFINITION. We *write arithmetically* when we repreſent by figures and ſigns any number and arithmetical propoſitions expreſſed in words.

Rule to write Numbers.

Beginning at the left hand, and writing towards the right, put every figure in ſuch place and period as the verbal expreſſion points out, ſupplying the omitted places with cyphers. Examples follow.

Ex-

Examples in Whole Numbers.

		Millions. $\overline{C}\,\overline{X}\,M$, C X U	:	Units. $\overline{C}\,\overline{X}\,M$, C X U
Nine hundred and eighty seven millions	is written thus,	9 8 7	:	0 0 0 , 0 0 0
Forty-five millions and seven hundred thousand		4 5	:	7 0 0 , 0 0 0
Six millions, four hundred and thirty-two thousand		6	:	4 3 2 , 0 0 0
Eight hundred and five thousand and nine hundred				8 0 5 , 9 0 0
Seventy-three thousand and ten				7 3 , 0 1 0
Five thousand and four				5 , 0 0 4
Four hundred and twenty				4 2 0
Ninety-five				9 5
Seven				7

Examples in Decimals.

		Units.	Tenths.	Hundredths.	Thousandths.	X Thousandths.	C Thousandths.	Millionths.
Nine units and five tenths	is written thus,	9	·5					
Forty-eight hundredths			·4	8				
Nine hundred and seven thousandths			·9	0	7			
Four hundred and nine ten-thousandths			·0	4	0	9		
Five hundred-thousandths			·0	0	0	0	5	
Seven hundred and three thousand and forty-six millionths			·7	0	3	0	4	6

Rule

Rule to write Fractions.

Write the figures which express the numerator of the given fraction above a line, and below it the figures of its denominator.

Thus, an half, two thirds, three quarters, seven tenths, &c. are written $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{10}$, &c.

Rule to write arithmetical Propositions.

It is only required to write down by figures and signs the numbers, their relations, and the operations, which are expressed in words, as is to be seen in the following examples:

Simple propositions.

I. The sum of the odd numbers 3 and 5 makes the product of the even numbers 2 and 4.	is written thus,	$3+5=2\times4$
II. The sum of the numbers 10 and 6 is four times their difference.		$10+6=4\times\overline{10-6}$
III. The product of 3 multiplied by 4 is equal to the quotient of 48 divided by four.		$3\times4=48\div4$
IV. The double of 3 is an half of 12.		$2\times3=\frac{12}{2}$
V. Take an half of the product of 3 muliplied by 4, add 8, and you will find the double of 3 and 4.		$\frac{3\times4}{2}+8=2\times\overline{3+4}$

Com-

Compound Propositions.

	is written thus,
I. The sum of six dozen dozen and half a dozen dozen is 936, and the difference 792.	$6 \times 12 \times 12 + \frac{12 \times 12}{2} = 936$ $6 \times 12 \times 12 - \frac{12 \times 12}{2} = 792$
II. The continual multiplication of nine digits will give 362880, the changes that may be rung on nine bells.	$1.2.3.4.5.6.7.8.9 = 362880$
III. If 8 cannons in one day spend 48 barrels of powder, 24 cannons in 22 days will spend 3168 barrels of it.	$8 \times 1 : 48 :: 24 \times 22 : 3168$
IV. A schoolmaster being asked how many scholars he had, said, I have 36; but if I had as many more, half as many, and 1 quarter as many, plus 1 scholar, I should have just 100.	$36 + 36 + \frac{36}{2} + \frac{36}{4} + 1 = 100$
V. A captain and 160 sailors took a prize worth 1360*l.* of which the captain has a fifth part for his share, and the rest having been equally divided among the sailors, each man's part was 6 pounds and eight tenths.	Captain's share $= 1360 \div 5$ The remainder $= 1360 - 1360 \div 5$ Each man's part $= \frac{1360 - 1360 \div 5}{160}$ Therefore $\frac{1360 - 1360 \div 5}{160} = 6.8l.$

CHAP.

CHAP. III.

Of Reading Arithmetically.

DEFINITION. We *read arithmetically* when we express in words any number given in figures, and also arithmetical propositions represented by figures and signs.

Rule to read Numbers.

Divide the given numbers in their periods and half-periods; then beginning at the left hand, and reading towards the right; to the simple value of every figure join the name of its place, and conclude each period by expressing its title, every where omitting the cyphers. Examples follow.

Examples in Whole Numbers.

Millions. Units. CXM,C X U : C X M , C X U		
9 0 0 : 0 0 0 , 0 0 0	is read thus,	Nine hundred millions.
7 8 : 0 0 9 , 0 0 0		Seventy-eight millions and nine thousand.
4 : 0 0 0 , 8 0 0		Four millions and eight hundred.
9 0 3 , 0 0 5		Nine hundred and three thousand and five.
2 0 , 0 3 4		Twenty thousand and thirty four.
8 , 4 6 9		Eight thousand four hundred and sixty nine.
7 0 8		Seven hundred and eight.
9 6		Ninety-six.
3		Three, or three units.

Note.

Note. The name of the unit's place, in all periods, and the title of the right-hand period, are commonly omitted, or very rarely expreſſed.

Examples in Decimals.

Units.	Tenths.	Hundredths.	Thouſandths.	X Thouſandths.	C Thouſandths.	Millionths.		
7	·5						is read thus,	Seven units and five tenths, or ſeventy-five tenths.
	·7	5						Seventy-five hundredths.
	.0	7	5					Seventy-five thouſandths.
	·7	0	5					Seven hundred and five thouſandths.
	·7	5	6	8				Seven thouſand five hundred and ſixty-eight ten-thouſandths.
	.0	7	0	5	6			Seven thouſand and fifty ſix hundred-thouſandths.
	·4	0	7	0	6	5		Four hundred and ſeven thouſand and ſixty-five millionths.

Note. Decimals may be reſolved into conſtituent parts, and the parts may be read ſeparately thus:

$.75 = .7 + .05$; that is, ſeven primes and five ſeconds.

$.075 = .07 + .005$; that is, ſeven ſeconds and five thirds.

$.705 = .7 + .005$; that is, ſeven primes and five thirds.

$.7568 = .7 + .05 + .006 + .0008$; that is, ſeven primes, five ſeconds, ſix thirds, and eight fourths.

Rule to read Fractions.

Read the numerator of the given fraction as a whole number, and its denominator as the ordinal numbers. Examples follow.

$\frac{1}{2}$	is read thus,	An half.
$\frac{2}{3}$		Two thirds or third parts.
$\frac{3}{4}$		Three fourths, or three quarters, or three fourth parts.
$\frac{7}{10}$		Seven tenths or tenth parts.
$\frac{75}{100}$		Seventy-five hundredths or hundredth parts.
$\frac{15}{7}$		Fifteen ſevenths or ſeventh parts.

Rule to read Arithmetical Propoſitions.

It is only required to expreſs by words the figures and ſigns with which the arithmetical propoſitions are written down; as is to be ſeen in the following examples;

$\overline{8+6}\times\overline{8-6}=8^2-6^2$	is read thus,	The ſum of 8 and 6 multiplied by their difference, is equal to the difference of their ſquares.
$1+3+5+7=4^2$		The ſum of the four odd numbers 1, 3, 5, 7, is equal to the ſquare of 4.
$1+3+5+7=4\times4$		The ſum of the four odd numbers 1, 3, 5, 7, is four times four.

The

$2+4+6+8=5\times 4$	The ſum of the four even numbers 2, 4, 6, 8, is five times four.
$8^2-6^2=8+6+2\times 7$	The difference of the ſquares of the numbers 8 and 6 gives the ſum of their roots, together with twice the number in the natural ſcale between the two roots.
$8^2-5^2=8+5+2\times\overline{6+7}$	The difference of the ſquares of the numbers 8 and 5 gives the ſum of their roots, together with twice the numbers in the natural ſcale between the two roots.

CHAP. IV.

Of Speaking Arithmetically.

THE proper object which is treated of in the arithmetical language being number, every propoſition derived from the arithmetical alphabet, the mathematical ſigns and the moſt ſimple reaſoning concerning the nature of numbers and their relations, muſt be reckoned arithmetical. Hence the following

Definition. We *speak arithmetically* when employing the arithmetical alphabet and the mathematical signs, no less in the investigation and demonstration of theorems, than in the resolution of problems, our reasoning and operations entirely depend on arithmetical principles, and proceed without any foreign aid or direction.

General Rule to find Theorems.

Write numbers arithmetically, consider their kind, form, position, known properties and relations, sum, difference, product; quotient, powers, roots, &c. and either marking the several operations by their signs to retain the original form of numbers, or doing these operations to change that form, you will gather different expressions, and consequently by reading them, discover some propositions, to be reckoned new arithmetical principles.

Example.

I propose to find some properties of the odd and even numbers, for instance, nine and three, twelve and four. Writing therefore these numbers, and marking their sum, difference, product, and quotient, I find the following expressions:

$9+3=12$	$12+4=16$	$9+4=13$	$12+3=15$
$9-3=6$	$12-4=8$	$9-4=5$	$12-3=9$
$9\times3=27$	$12\times4=48$	$9\times4=36$	$12\times3=36$
$9\div3=3$	$12\div4=3$	$9\div4=2\frac{1}{4}$	$12\div3=4$

Now considering the kinds of these numbers, and reading these expressions, I learn the following propositions.

I. The sum or difference of two odd numbers is an even number.

II. The

II. The sum or difference of two even numbers is an even number.

III. The sum or difference of an odd and an even number is an odd number.

IV. The product of two odd numbers is an odd number.

V. The product of two numbers, either even or odd and even, is an even number.

VI. The whole quotient of two odd numbers is an odd number.

VII. The whole quotient of two even numbers may be an odd number.

VIII. The quotient arising from the division of an odd number by an even one, cannot be a whole number.

IX. The whole quotient found by the division of an even number by an odd one must be an even number.

General Rule to resolve Problems by Reasoning.

Endeavour to form a clear idea of the question proposed, and cutting off what is foreign to the nature of numbers, set down in few words the arithmetical proposition: then, considering the properties of the given numbers, the conditions of the problem, that is, the relations of known and unknown numbers, the several operations to be done, &c. draw the consequences contained in them; and pass from one consequence to another, until you arrive at a proposition manifest enough to give the answer required.

Example.

A certain captain sends out 100 soldiers, divided into two bands, whose difference was 20 men; how many soldiers were in the greater, and how many in the less band?

Resolution. The simple arithmetical question contained in this problem, is but the following.

To

To find two numbers, whose sum is 100, and difference 20.

Because the greater number surpasses the less by 20 units, the greater wanting 20 units must be equal to the less, or the less increased by 20 units, will be equal to the greater.

In the first case, twice the greater number, minus 20 units, being equal to the given sum 100, once the greater number wanting 10 units, will be half this sum, or 50; and consequently, the greater number must be 50 increased by 10 units, that is, half the sum and half the difference, or 60.

In the second case, twice the less number, plus 20 units, being also equal to the given sum 100, once the less number increased by 10 units, will be half this sum, or 50; and consequently, the less number must be 50 wanting 10 units, that is, half the sum, wanting half the difference, or 40.

Hence the general answer; 1st, *Add the difference to the sum, and half the aggregate will be the greater number*; 2dly, *subtract the difference from the sum, and half the remainder will be the less number.*

General Rule to resolve Problems by Trials.

Take any number at pleasure, and by it working the question according to the nature thereof as if it was the true number, bring out the result, which, if not the true, will serve at least to correct your supposition, and to find another result, by whose comparison with the first, it will not be difficult to contrive some particular way, in order to discover the true number sought.

Example.

Good-morrow, good fellow, with your 20 geese: nay, said he, I have not 20; but if I had as many, half

as

as many, 2 geese and $\frac{1}{2}$, then I should have 20. I demand how many he had?

Suppose 3; then as many, $\frac{1}{2}$ as many, 2 geese and $\frac{1}{2}$, would make $3+3+1\frac{1}{2}+2\frac{1}{2}=10$, which should be 20. The error therefore is $20-10=10$.

Again, suppose he had 5; then as many, $\frac{1}{2}$ as many, 2 geese and $\frac{1}{2}$, would make $5+5+2\frac{1}{2}+2\frac{1}{2}=15$, which should be 20. The error therefore is $20-15=5$.

Whence it appears, that the first supposition 3 having been increased by two units, the first error is diminished by 5, and then increasing the second supposition by two units, the remaining error 5 will be perhaps taken away. Really we find $7+7+3\frac{1}{2}+2\frac{1}{2}=20$, as it must be.

From these general rules may be drawn particular ones of great use in the arithmetical language, as will be seen in the following Chapters.

CHAP. V.

The Rule of Definitions, wherein of the Principal Operations.

THE Rule of Definitions is a method of finding what is required by way of its definition and the arithmetical principles.

Rule of Definitions. Endeavour to define clearly what is to be found, set down the arithmetical principles belonging to the present question, and hence draw the rules to resolve it.

We will now apply this rule to find the particular methods of doing the principal operations upon the whole

whole and broken numbers; and therefore we begin with setting down the following arithmetical

Principles. I. None but similar or like things can be added or subtracted, viz. units and units, tens and tens, hundreds and hundreds, &c. tenths and tenths, hundredths and hundredths, &c. thirds and thirds, fourths and fourths, &c. and in general, numbers of the same kinds and denominations.

II. The figures of any number increase in value from the right toward the left, in a decuple proportion.

III. Ten in any inferior place makes one or an unit in the next higher place.

IV. If the right-hand figure of any number be cut off, the remaining figure or figures are a just number of tens, and the right-hand figure so cut off is the overplus. Thus, cutting off 2 from 72, the remaining figure 7 is seven tens, cutting off 0 from 720, the remaining figures 72 are seventy-two tens.

V. Numbers equally augmented or diminished continue to have the same difference. Thus, because $8+4=12$, and $6+4=10$, it will be $8-6=12-10=2$. Again, being $8-4=4$, and $6-4=2$, it will be $8-6=4-2=2$.

VI. Any number is naturally resolved into as many constituent parts as it has significant figures, by annexing to each significant figure as many cyphers as there are figures on its right hand. Thus, $345=300+40+5$; $6082=6000+80+2$; $70090=70000+90$.

VII. Any whole is equal to all its parts.

VIII. The difference of two unequal numbers added to the less, gives a sum equal to the greater; or subtracted from the greater, leaves a remainder equal to the less. Thus, because $8-5=3$, it is $5+3=8$, and $8-3=5$.

IX. If the numerator and denominator of a fraction be either both multiplied or both divided by the same num-

number, the products or quotients will retain the same proportion to one another; and consequently the new fraction thence arising will be of the same value with the given one. Thus the numerator and denominator of the fraction $\frac{2}{4}$ multiplied by 2 produces $\frac{4}{8}$, and divided by 2 quotes $\frac{1}{2}$, both which fractions are of the same value with $\frac{2}{4}$.

SECTION I.

ADDITION.

DEFINITION. *Addition* is the collecting of two or more numbers into a *sum* or *total*.

I. *Addition of Integers.*

RULES. I. *Set figures of like place under each other, viz. units under units, tens under tens, hundreds under hundreds, &c.* See Princ. I.

II. *Draw a line under the lowest number; then beginning at the lowest place, set down the right-hand figure of the sum of every column, and carry the rest as so many units to the next superior place.* See Principles II. III. IV.

Example.

Having placed the numbers as directed in Rule I. viz. units under units, &c. as in the margin, and beginning at the lowest place, viz. that of units, I say, 2 and 1 make 3, and 3 make 6, and 4 make 10; which being just 1 ten, and nothing over,

5974	
9803	
7541	
862	
	+
24180	

 I set

I set down the right-hand figure 0 in the place of units, and, because ten in any lower place makes but one in the next superior place, I carry my one ten, as directed in Rule II. saying 1 ten, collected out of the units, and 6 tens make 7 tens, and 4 make 11, and 0 makes but still 11, and 7 make 18; here again I set down the right-hand figure 8 and carry the remaining figure 1, being 1 hundred, to the next place, viz. that of hundreds; and having in like manner added up this column, the amount is 31; so I set down the right-hand figure 1 in the place of hundreds; and carry the remaining figure 3 to the next place or column, which being also added, amounts to 24; I set the right-hand figure 4 below in its proper place, and the remaining figure 2, which belongs to the next place, I set on the left hand, there being no figure in the next place to which it can be carried. So the sum or total is 24180.

The reason of the operation will still farther appear by taking the sum of each column separately, and then adding them into one total, as in the margin.

	5974	
	9803	
	7541	
	862	
	———	+ sign of Addition.
Sum of the	10	units.
	17 .	tens.
	30 . .	hundreds.
	21 . . .	thousands.
	———	
	24180	total.

II. *Addition of Decimals.*

RULE. *Place the given decimals so that the points may stand directly under each other, and consequently tenths under tenths, hundredths under hundredths, &c. then add them as integers, inserting the decimal point directly under the column of points.*

Ex. 1.			Ex. 2.		
.75	=	.750	276.	=	276.0000
.895	=	.895	39.213	=	39.2130
.5	=	.500	724.07	=	724.0700
.625	=	.625	450.9	=	450.9000
.725	=	.725 +	0.3785	=	.3785 +
3.495	=	3.495	1490.5615	=	1490.5615

III. *Addition of Fractions.*

RULES. I. *If the given fractions have all the same denominator, add the numerators, and place the sum over the denominator.*

II. *If the given fractions have different denominators, reduce them to a common denominator by the following Lemma, then work as in* Rule I.

LEMMA. *To reduce fractions of different denominators to a common denominator, multiply the denominators continually for the common denominator, and multiply each numerator into all the denominators, except its own, for the several numerators.*

Examples.

I. Reduce $\frac{3}{4}$ and $\frac{4}{5}$ to a common denominator.

$4 \times 5 = 20$, the common denominator. Or, $\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}$.
$3 \times 5 = 15$, the firſt numerator.

$4 \times 4 = 16$, the ſecond numerator. $\frac{4}{5} = \frac{4 \times 4}{5 \times 4} = \frac{16}{20}$.

So the new fractions are $\frac{15}{20}$ and $\frac{16}{20}$.

II. Reduce $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{8}$, to a common denominator.

$3.6.8 = 144$, common denominator. Or, $\frac{2}{3} = \frac{2.6.8}{3.6.8} = \frac{96}{144}$
$2.6.8 = 96$, firſt numerator.

$5.3.8 = 120$, ſecond numerator. $\frac{5}{6} = \frac{5.3.8}{6.3.8} = \frac{120}{144}$

$7.3.6 = 126$, third numerator. $\frac{7}{8} = \frac{7.3.6}{8.3.6} = \frac{126}{144}$

The new fractions are $\frac{96}{144}$, $\frac{120}{144}$, and $\frac{126}{144}$.

III. Reduce $4 = \frac{4}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$ to a common denominator.

$1.2.3.5 = 30$, common denom. Or, $\frac{4}{1} = \frac{4.2.3.5}{1.2.3.5} = \frac{120}{30}$
$4.2.3.5 = 120$, firſt numerator.
$1.1.3.5 = 15$, ſecond numerator. $\frac{1}{2} = \frac{1.1.3.5}{2.1.3.5} = \frac{15}{30}$
$2.1.2.5 = 20$, third numerator.
$3.1.2.3 = 18$, fourth numerator.

$\frac{2}{3} = \frac{2.1.2.5}{3.1.2.5} = \frac{20}{30}$

New fractions $\frac{120}{30}$, $\frac{15}{30}$, $\frac{20}{30}$, and $\frac{18}{30}$.

$\frac{3}{5} = \frac{3.1.2.3}{5.1.2.3} = \frac{18}{30}$

Notes. 1. The ſecond method, ſubjoined in every example, to reduce fractions of different denominators to equivalent fractions of a common one, may ſerve alſo

alſo as a demonſtration of the firſt. See Principle IX.

2. In the third example we conſider the whole number 4 as a fraction whoſe denominator is 1, to apply to it the Rules of the broken numbers.

Examples of Addition.

I. What is the ſum of $\frac{2}{7}$ and $\frac{3}{7}$?

$$\text{Anſ. } \frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$$

II. What is the ſum of $\frac{2}{5}$, and $\frac{3}{8}$?

$$\text{Anſ. } \frac{2}{5}+\frac{3}{8}=\frac{2\times 8}{5\times 8}+\frac{3\times 5}{8\times 5}=\frac{16}{40}+\frac{15}{40}=\frac{31}{40}$$

III. What is the ſum of 3, $\frac{2}{5}$, and $\frac{4}{7}$?

$$\text{Anſ. } \frac{3}{1}+\frac{2}{5}+\frac{4}{7}=\frac{3\times 5\times 7+2\times 1\times 7+4\times 1\times 5}{1\times 5\times 7}=\frac{105+14+20}{35}=\frac{139}{35}$$

IV. *Addition of ſeveral Denominations.*

DEFINITION. Addition of ſeveral Denominations is the finding of the ſum of ſeveral parts of integers, ſuch as ſhillings, pence, farthings; ounces, &c.

RULES. I. *Place like parts under each other; viz. farthings under farthings, pence under pence, &c.* See Principle I.

II. *Begin at the loweſt of the parts, add every column as in integers, and carry according to the value of an unit of the next ſuperior denomination; viz. for every four in the ſum of farthings carry* 1 *to the pence, for every twelve in the pence carry* 1 *to the ſhillings, and for every twenty in the ſhillings carry* 1 *to the pounds.*

TA-

TABLES.

I. Money.

4 farthings make	1 penny.
12 pence	1 shilling.
20 shillings	1 pound.

Marked thus.

l.	*s.*	*d.*	*f.* or *q.*
1=	20=	240=	960
	1=	12=	48
		1=	4

II. Avoirdupois Weight.

16 drams make	1 ounce.
16 ounces	1 pound.
28 pounds	1 quarter.
4 quarters	1 hundred.
20 hundreds	1 tun.

Marked thus.

t.	*c.*	*q.*	*lb.*	*oz.*	*dr.*
1=	20=	80=	2240=	35840=	573440
	1=	4=	112=	1792=	28672
		1=	28=	448=	7168
			1=	16=	256
				1=	16

Examples.

Examples.

I.

	(10) *l.*	(20) *s.*	(12) *d.*	(4) *f.*
	74	18	11	3
	96	9	10	2
	58	17	8	1
	63	11	9	2
A.	291	55	38	8
B.	291	55	40	0
C.	291	58	4	0
D.	293	18	4	0

II.

	(10) *t.*	(20) *c.*	(4) *q.*	(28) *lb.*	(16) *oz.*	(16) *dr.*
	74	19	3	27	15	12
	85	17	2	24	14	10
A.	159	36	5	51	29	22
B.	159	36	5	51	30	6
C.	159	36	5	52	14	6
D.	159	36	6	24	14	6
E.	159	37	2	24	14	6
F.	160	17	2	24	14	6

Note. In theſe examples the firſt number A gives the ſum of every denomination, ſeparately, without any reduction; the ſecond B ſhews the ſame ſum, but with the ſuitable reduction of the loweſt of the parts, according to the value of an unit of the next ſuperior denomination; and ſo continually the next following number expreſſes the ſame ſum, with one reduction more, ſo that the laſt number, viz. D in the firſt example, and F in the ſecond, repreſents the total entirely reduced.

SEC-

SECTION II.

SUBTRACTION.

DEFINITION.—*Subtraction* is the taking a less number from a greater, in order to discover their *difference*, or the *remainder*.

I. *Subtraction of Integers.*

RULES. I. *Set figures of like place under each other, viz. units under units, tens under tens, &c. and the greater of the given numbers uppermost.* Princ. I.

II. *Beginning at the place of units, take the lower figures from those above, borrowing and paying ten, as need requires, and write the remainders below a line drawn under the lowest number.* Princ. II. III. V.

Example.

Having placed the numbers as directed in Rule I. viz. units under units, &c. as in the margin, and beginning at the place of units, I say 3 units from 6 units, and 3 units remain, which I set below in the place of units; then 5 tens from 5 tens, and nothing remains, wherefore I set 0 below in the place of tens; I proceed and say 8 hundreds from 3 hundreds I cannot, but, because an unit in the next superior place makes ten in this place, I borrow 1, i. e. 1 ten, from the said next place, as

74356	major or minuend.
27853	minor or subtrahend.
———	—sign of Subtraction.
46503	difference or rem.

as directed in Rule II. which 1 ten being added to 3 makes 13; then I say 8 from 13, and 5 remains, which 5 I set below in the place of hundreds; then I proceed, and pay the unit borrowed, either by estimating 4, the next figure in the major, to be only 3, or, which is more usual, and the same in effect, by adding 1 to the next figure in the minor, thus, 1 that I borrowed and 7 make 8, from 4 I cannot, but borrowing as before, I say 8 from 14 and 6 remains, which 6 I set below; I go on, and say 1 borrowed and 2 make 3, from 7, and 4 remains, which 4 I set below: so the difference or remainder is 46503.

The reason of the operation will still farther appear by dividing the major number 74356 into two constituent parts 13000 and 61356, and then subtracting from them the minor, viz. 3 units from 6 units, 5 tens from 5 tens, 8 hundreds from 13 hundreds, 7 thousands from 13 thousands, and lastly 2 ten-thousands from 6 ten-thousands, as in the margin.

13000	major.
61356	
27853	minor.
46503	rem.

II. *Subtraction of Decimals.*

RULE. *Place the minor under the major, so that the points may be in one column, and then work as in Subtraction of Integers, putting the decimal point in the difference directly under the points.*

Ex. 1.		Ex. 2.	
From	.87 =.8700	From	45.9380003
Sub.	.3894=.3894	Sub.	3.9857986
Rem.	.4806=.4806	Rem.	41.9522017

III. *Subtraction of Fractions.*

RULE. *Reduce all the fractions to the ſame denominator if they be not, then the difference of the numerators written above the common denominator will be the difference of the fractions required.*

Examples.

I. From $\frac{5}{7}$ ſubtract $\frac{3}{7}$. Anſ. $\frac{5}{7}-\frac{3}{7}=\frac{2}{7}$.

II. From $\frac{3}{4}$ ſubtract $\frac{2}{3}$.

$$\text{Anſ. } \frac{3}{4}-\frac{2}{3}=\frac{3.3}{4.3}-\frac{2.4}{3.4}=\frac{9}{12}-\frac{8}{12}=\frac{1}{12}$$

III. From $\frac{9}{10}$ ſubtract $\frac{1}{2}$ and $\frac{1}{3}$.

$$\text{Anſ. } \frac{9}{10}-\frac{1}{2}-\frac{1}{3}=\frac{9.2.3}{10.2.3}-\frac{1.10.3}{2.10.3}-\frac{1.10.2}{3.10.2}=$$

$$\frac{54}{60}-\frac{30}{60}-\frac{20}{60}=\frac{4}{60}$$

IV. From 2 and $\frac{3}{4}$ ſubtract $\frac{2}{3}$ and $\frac{5}{6}$.

$$\text{Anſ. } \frac{2}{1}+\frac{3}{4}-\frac{2}{3}-\frac{5}{6}=\frac{2.4.3.6}{1.4.3.6}+\frac{3.1.3.6}{4.1.3.6}-\frac{2.1.4.6}{3.1.4.6}$$

$$-\frac{5.1.4.3}{6.1.4.3}=\frac{144}{72}+\frac{54}{72}-\frac{48}{72}-\frac{60}{72}=\frac{198}{72}-\frac{108}{72}=\frac{90}{72}$$

V. From $7\frac{3}{4}$ ſubtract $5\frac{1}{2}$.

$$\text{Anſ. } \frac{7}{1}+\frac{3}{4}-\frac{5}{1}-\frac{1}{2}=\frac{7.4.1.2}{1.4.1.2}+\frac{3.1.1.2}{4.1.1.2}-\frac{5.1.4.2}{1.1.4.2}$$

$$-\frac{1.1.4.1}{2.1.4.1}=\frac{56}{8}+\frac{6}{8}-\frac{40}{8}-\frac{4}{8}=\frac{62}{8}-\frac{44}{8}=\frac{18}{8}$$

IV. *Subtraction of several Denominations.*

RULES. I. *Write the less number under the greater, as directed in Addition of several Denominations.*

II. *Then, beginning at the least denomination, subtract the under number of each from the upper, borrowing and paying according to the value of an unit of the next superior denomination, as need requires.*

Examples.

I.

	(10) *l.*	(20) *s.*	(12) *d.*	(4) *f.*	
	73	15	10	2	maj.
	48	12	6	2	min.
	25	3	4		rem.

II.

	(10) *l.*	(20) *s.*	(12) *d.*	(4) *f.*	
A.	708	14	6	1	maj.
B.	707	33	17	5	maj.
	278	17	10	3	min.
	429	16	7	2	rem.

III.

	(10) *t.*	(20) *c.*	(4) *q.*	(28) *lb.*	(16) *oz.*	(16) *dr.*	
A.	38	14	2	18	13	8	maj.
B.	37	33	6	17	29	8	maj.
	25	14	3	12	15	6	min.
	12	19	3	5	14	2	rem.

Note. In the 2d and 3d examples the major number A is changed into its equal B, according to the value of an unit of the next superior denomination, in order to render every part of it greater than the correspondent part in the minor, and thus the method of Subtraction is made obvious to sight.

SECTION III.

MULTIPLICATION.

IN Multiplication there are two numbers given, viz. one to be multiplied, called the *multiplicand*, and another that multiplies it, called the *multiplier*; these two go under the common names of *factors*, and the number arising from the multiplication of the one by the other is called the *product*, and sometimes the *fact*, or the *rectangle*. If a multiplier consists of two or more figures, the numbers arising from the multiplication of these several figures into the multiplicand are called *particular* or *partial products*, and their sum is called the *total product*.

DEFINITION. Multiplication then is the taking or repeating of the multiplicand as often as the multiplier contains unity. Or,

Multiplication, from a multiplicand and a multiplier given, finds a third number, called the product, which contains the multiplicand as often as the multiplicand contains unity.

Hence Multiplication supplies the place of many Additions; for if the multiplicand be repeated, or set down as often as there are units in the multiplier, the sum of these taken by Addition will be equal to the product by Multiplication. Thus $5 \times 3 = 15 = 5 + 5 + 5$.

The first and lowest step in Multiplication is, to multiply one digit by another; and the fact or number thence arising is called a *single product*. This elementary step may be learned from the following Table commonly called *Pythagoras's Table of Multiplication*: which is

is consulted thus: seek one of the digits or numbers on the head, and the other on the left side, and in the angle of meeting you have their product. The learner before he proceed farther ought to get the table by heart. To Pythagoras's table are here added on account of their usefulness the products of the numbers 10, 11, 12.

TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

I. *Multiplication of Integers.*

RULES. I. *Set the multiplier below the multiplicand, so as like places may stand under each other, viz. units under units, tens under tens, &c. but if either or both of the factors have cyphers on the right hand, set their first significant figures under each other.*

II. *Beginning at the right hand, multiply each figure of the multiplier into the whole multiplicand, carrying as in Addition, and placing the right-hand figure of each particular product directly under the multiplying figure.* Princ. II. III. IV. VI.

III. *Add the particular products, and their sum will be the total-product.* Princ. VII.

Note. The order prescribed in Rule I. is not absolutely necessary, but very convenient, as will appear in the examples.

Examples.

742	multiplicand.
68	multiplier.
———	× sign of Mult.
5936	particular products.
4452	
———	+ sign of Addit.
50456	total product.

Having placed the multiplier under the multiplicand, as directed in Rule I. I proceed to the operation, and say 8 times 2 make 16; I set the 6 below in the place of units, and carry my 1 ten to the next place, as directed in Rule II. saying 8 times 4 make 32, and 1 that I carried make 33; I set the 3 tens below in the place of tens, and carry the 3 hundreds to the next place; then I proceed and say 8 times 7 make 56, and 3 that I carried make 59; I set 9 below in the place of hundreds, and the 5, which belongs to the next place, I set on its left hand, there being no farther place to which it can

can be carried. Then I proceed, and multiply 6 tens into the whole multiplicand, and I ſet down the right-hand figure of the product under the multiplying figure, or in the place of tens, and the other figures in the the next following places, as in the margin. Laſtly, adding the two particular products, the total product will be 50456.

If you ſet down in its proper place ſeparately every ſingle product ariſing from the multiplication of each figure of the multiplier into the multiplicand, as in Examples I. II. III. IV. you will ſtill further underſtand the general method of Multiplication and its reaſon.

I.

	742	multiplicand.
	68	multiplier.
	—— ×	
Sum of the ſingle products	16	units.
	32 .	tens.
	56 . .	hundreds.
	12 .	tens.
	24 . .	hundreds.
	42 . . .	thouſands.
	—— +	
	50456	total product.

II.	III.	IV.
853	853000	853000
72000	72	7200
—— ×	—— ×	—— ×
106	106	106
16 . .	16 . .	16 . .
21 .	21 .	21 .
35 . .	35 . .	35 . .
56 . . .	56 . . .	56 . . .
—— +	—— +	—— +
61416000	61416000	6141600000

V.

V.

29601847	multiplicand.
300905	multiplier.
	×
148009235	single products.
266416623 . .	
88805541	
	+
8907343771535	total product.

VI.

29601847	multiplicand.
5	partial multipliers.
900	
300000	
	×
148009235	product by 5.
26641662300	product by 900.
8880554100000	product by 300000.
	+
8907343771535	total product.

The reason of setting the right-hand figure of each particular product directly under the multiplying figure will still farther appear by resolving the multiplier into its constituent parts, as may be seen in Ex. VI. which is a repetition of Ex. V.

Note. If the multiplicand or multiplier, or both end with cyphers, it is enough to annex to the total product as many cyphers as there are annexed to them both; as in Ex. II. III. IV.

II. *Mul-*

II. *Multiplication of Decimals.*

RULE. *Work exactly as in Multiplication of Integers, placing the decimal point in the product so as to make just as many decimals in it as there are in both factors; and if the product has not so many figures, supply that defect by prefixing cyphers.*

Notes. I. The reason of this Rule will easily appear by considering, that the multiplicand becomes ten times, hundred times, &c. less, as it is multiplied by tenths, by hundredths, &c.

II. In the following Examples II. and III. the product not affording so many decimal figures as are in the multiplicand and multiplier, I supply the defect by prefixing cyphers.

Ex. I.	Ex. II.
·785	·125
75 ×	25 ×
3925	625
5495 +	250 +
·58875	·03125

Ex. III.	Ex. IV.
·0375	6.875
·105 ×	·25 ×
1875	34375
375 +	13750 +
·0039375	1·71875

III. *Multiplication of Fractions.*

RULE. *Multiply the numerators for the numerator of the product, and multiply the denominators for its denominator.*

Note. The reason of the Rule may be shewn thus: $\frac{2}{3}\times\frac{4}{5}=\frac{2\times4}{3\times5}=\frac{8}{15}$: for 2 times $\frac{4}{5}=\frac{2\times4}{5}=\frac{3\times2\times4}{3\times5}$ (see Principle IX.), and therefore $\frac{2}{3}$ times $\frac{4}{5}$ will be $=\frac{3\times2\times4}{3\times5}$ divided by 3, that is $\frac{3\times2\times4\div3}{3\times5}=\frac{2\times4}{3\times5}$: hence $\frac{2}{3}\times\frac{4}{5}=\frac{2\times4}{3\times5}=\frac{8}{15}$.

Examples.

I. $\frac{2}{3}\times\frac{5}{7}=\frac{2\times5}{3\times7}=\frac{10}{21}$

II. $\frac{2}{3}\times\frac{3}{4}=\frac{2\times3}{3\times4}=\frac{2}{4}=\frac{2\times1}{2\times2}=\frac{1}{2}$. See Principle IX.

III. $\frac{2}{3}\times\frac{5}{7}\times\frac{6}{10}=\frac{2.5.6}{3.7.10}=\frac{2.5.2.3}{3.7.2.5}=\frac{2}{7}$

IV. $\frac{3}{4}\times5\frac{2}{3}=\frac{3}{4}\times\overline{\frac{5}{1}+\frac{2}{3}}=\frac{3}{4}\times\frac{\overline{15+2}}{3}=\frac{3}{4}\times\frac{17}{3}=\frac{17}{4}=4\frac{1}{4}$

V. $2\frac{1}{4}\times5\frac{1}{3}=\overline{\frac{2}{1}+\frac{1}{4}}\times\overline{\frac{5}{1}+\frac{1}{3}}=\frac{8+1}{4}\times\frac{15+1}{3}=\frac{9}{4}\times\frac{16}{3}=3\times4=12$

VI.

VI. $\frac{785}{1000} \times \frac{75}{100} = \frac{785 \times 75}{1000 \times 100} = \frac{58875}{100000} = .58875$

VII. $\frac{125}{1000} \times \frac{25}{100} = \frac{125 \times 25}{1000 \times 100} = \frac{3125}{100000} = .03125$

Notes. I. The second and third examples shew, that in multiplying fractions, equal factors above and below may be dashed or dropped, and that a factor above and another below may be divided by the same number, or that we may change one numerator for another: thus $\frac{2}{3} \times \frac{3}{4} = \frac{3}{3} \times \frac{2}{4} = \frac{3}{3} \times \frac{2 \div 2}{4 \div 2} = \frac{1}{1} \times \frac{1}{2} = \frac{1}{2}$; likewise $\frac{2}{3} \times \frac{5}{7} \times \frac{6}{10} = \frac{6}{3} \times \frac{5}{7} \times \frac{2}{10} = \frac{6}{3} \times \frac{1}{7} \times \frac{10}{10} = \frac{6 \div 3}{3 \div 3} \times \frac{1}{7} \times \frac{10}{10} = \frac{2}{1} \times \frac{1}{7} \times \frac{1}{1} = \frac{2}{7}$

II. In the fourth and fifth examples the multiplication of the mixt numbers is performed.

III. Lastly, the sixth and seventh examples express another reason of the given Rule of the multiplication of Decimals.

IV. *Multiplication of several Denominations.*

RULE I. *If your multiplier is an integer, multiply it into all the parts of the multiplicand, beginning at the lowest, and carrying always as in Addition, or according to the value of the next superior place.*

Example.

What is the price of 14 packs of cloth, at 64*l.* 8*s.* 10*d.*½ *per* pack?

Here I say, 14 times 2 is 28, which being exactly 7 pence, I carry 7 to the place of pence, saying 14 times 10 is 140, and 7 that I carried make 147,

l.	s.	d.	fr.	
64	8	10	2	multiplic.
			14	multiplier.
			×	
902	4	3	0	product.

which is 12 shillings and 3 pence; I set down the 3 pence, and carry 12 to the place of shillings, saying 14 times 8 is 112 and 12 that I carried is 124, which makes 6 pounds and 4 shillings; I set down the 4 shillings, and carry 6 to the place of pounds, which are integers.

RULE II. *If your multiplier is a fraction multiply its numerator into all the parts of the multiplicand, as in Rule I. then divide the product by the denominator of the given fraction, according to Rule I. of Division of several Denominations, in the following Section.*

Note. The method is very simple; but because it supposes the division of several denominations by an integer, the learner, before he proceed to examples, ought to be acquainted with this operation.

Example.

What is the price of $\frac{2}{3}$ of a yard of velvet, at 3*l.* 8*s.* 4*d.* *per* yard?

Here the multiplier being $\frac{2}{3}$, I first multiply the multiplicand 3*l.* 8*s.* 4*d.* by the numerator 2; and then I divide the product 6*l* 16*s.* 8*d.* by the denominator 3; so the quotient 2*l.* 5*s.* 6*d.* 2*f.* $\frac{2}{3}$ will give the answer.

l.	s.	d.	
3	8	4	multiplicand.
			$\frac{2}{3}$ multiplier.
			× 2
6	16	8	product by 2.
			÷ 3
2	5	6	2*f.* $\frac{2}{3}$ quotient by 3.

RULE

RULE III. *If your multiplier is a mixt number, first multiply by the integer* (Rule I.) *then by the fraction* (Rule II.) *and the sum of these two products is the answer.*

Examples.

What is the price of 5¾ yards of linen, at 3*s.* 4*d.* *per* yard?

s.	*d.*			*s.*	*d.*		*s.*	*d.*	
3	4	multiplicand.		3	4		3	4	
	5¾	multiplier.			5			¾	
		×							× 3
16	8	price of 5 yards.		16	8		10	0	
2	6	price of ¾ of a yard.							÷ 4
		+					2	6	
19	2	total price.							

RULE IV. *If your multiplier is of several denominations, write down the parts of it with their respective denominators, as fractions of integers; then work according to the preceding Rules.*

Example.

What is the price of 5 yards, 3 quarters, 3 nails of cloth at 1*l.* 7*s.* 4*d.* *per* yard?

l.	*s.*	*d.*	
1	7	4	
5	¾	$\frac{3}{16}$	
		×	
6	16	8	price of 5 yards.
1	0	6	price of ¾ of a yard.
	5	1½	price of $\frac{3}{16}$ of a yard.
8	2	3½	total price.

See the several operations in the following page.

l. s. d.	l. s. d.	l. s. d.
1 7 4	1 7 4	1 7 4
5 ×	3 ×	3 ×
6 16 8	4 2 0	4 2 0
	÷ 4	÷ 16
	1 0 6	5 1½

Because 1 yard is either 4 quarters or 16 nails, 3 quarters and 3 nails will make $\frac{3}{4}+\frac{3}{16}$ of a yard: our multiplier is then $5+\frac{3}{4}+\frac{3}{16}$ yards. Multiplying by $\frac{3}{16}$, there is a remainder of $\frac{8}{16}$ of a penny, that is, 2 farthings.

Note.—This multiplication may be also performed by the method of fractions, reducing both multiplicand and multiplier to the least respective denomination. Thus, because 1 pound is 240 pence, and 7 shillings make 84 pence, 1 pound, 7 shillings, and 4 pence will be together 328 pence, or $\frac{328}{240}$ of a pound; likewise, 5 yards being 80 nails, and 3 quarters 12 nails, 5 yards, 3 quarters, and 3 nails, will give together 95 nails or $\frac{95}{16}$ of a yard. Therefore, multiplying $\frac{328}{240}$ by $\frac{95}{16}$, or $\frac{41}{30}$ by $\frac{95}{16}$ (for $\frac{328}{240} = \frac{328\div 8}{240\div 8} = \frac{41}{30}$), it will be $\frac{95}{16} \times \frac{41}{30} = \frac{3895}{480} = 8\frac{55}{480}$ pounds. Now $\frac{55}{480} = \frac{55\div 2}{480\div 2} = \frac{27\frac{1}{2}}{240}$, that is, $27\frac{1}{2}$ pence, or 2 shillings, 3 pence, and 2 farthings. The total product is then 8*l.* 2*s.* 3*d.* 2*f.* as before.

Definitions. I. *Dimension* is a measure of extension.

II. *A lineal dimension* is a lineal measure of extension having only length, as lines.

III. *A superficial dimension* is a superficial measure of extension, having only length and breadth, as surfaces.

IV. *A square dimension* is a superficial dimension, whose length and breadth are equal, as a square.

V. *A*

V. *A solid dimension* is a solid measure of extension consisting of length, breadth, and thickness, as solids.

VI. *A cubic dimension* is a solid measure of extension, whose length, breadth, and thickness are of the same magnitude, as a cube.

Note. The dimensions are usually taken in yards, feet, inches, and lines, whose value is as in the following Tables:

I. Long Measure.

12 lines	make	1 inch.
12 inches		1 foot.
3 feet		1 yard.

II. Superficial Measure.

144 square lines	make	1 square inch.
144 square inches		1 square foot.
9 square feet		1 square yard.

III. Solid Measure.

1728 cubic lines	make	1 cubic inch.
1728 cubic inches		1 cubic foot.
27 cubic feet		1 cubic yard.

RULE V. *To multiply the lineal dimensions one by another to have a product of a square dimension, write down the parts both of the multiplicand and the multiplier with their respective denominators, as fractions of an integer; then work as in the preceding Rule IV. and lastly, reduce those parts of the product which are not reduced to the denomination of any square measure, either multiplying or dividing the fractions, by which they are expressed, by a denominator fit for that purpose, as may be seen in the following*

Example.

Example.

In an area or floor, in length 38 feet, 9 inches, 6 lines, and in breadth 23 feet, 8 inches, 6 lines, how many square feet?

f.	in.	li.	
38	9	6	$=38+\frac{9}{12}+\frac{6}{144}$ lineal multiplicand.
23	8	6	$=23+\frac{8}{12}+\frac{6}{144}$ lineal multiplier.

$$892+\frac{30}{144} \text{ superficial product by } 23.$$

$$25+\frac{124}{144} \text{ superfical product by } \tfrac{8}{12}.$$

$$1+\frac{88}{144}+\frac{108}{144.144} \text{ sup. prod. by } \tfrac{6}{144}$$

Square Measures.

f.	in.	li.	
919	98	108	$=919+\frac{98}{144}+\frac{108}{144.144}$ total sq. prod.

$$38+\tfrac{9}{12}+\tfrac{6}{144}$$
$$23$$

$$17+\tfrac{3}{12}+\tfrac{138}{144}$$
$$114$$
$$76$$

$$891+\tfrac{3}{12}+\tfrac{138}{144} \text{ or}$$
$$891+\tfrac{3.12}{12.12}+\tfrac{138}{144} \text{ or}$$
$$891+\tfrac{36}{144}+\tfrac{138}{144} \text{ or}$$
$$891+\tfrac{174}{144} \text{ or}$$
$$892+\tfrac{30}{144}$$

38

$$38 + \frac{9}{12} + \frac{6}{144}$$
$$\frac{8}{12}$$
$$\times$$
$$25 + \frac{4}{12} + \frac{72}{144} + \frac{48}{12.144} \text{ or}$$
$$25 + \frac{4 \cdot 12}{12.12} + \frac{72}{144} + \frac{48 \div 12}{\frac{12.144}{12}} \text{ or}$$
$$25 + \frac{48}{144} + \frac{72}{144} + \frac{4}{144} \text{ or}$$
$$25 + \frac{124}{144}$$

$$38 + \frac{9}{12} + \frac{6}{144}$$
$$\frac{6}{144}$$
$$\times$$
$$1 + \frac{84}{144} + \frac{54}{12.144} + \frac{36}{144.144} \text{ or}$$
$$1 + \frac{84}{144} + \frac{54 \div 12}{\frac{12.144}{12}} + \frac{36}{144.144} \text{ or}$$
$$1 \frac{84}{144} + \frac{4}{144} + \frac{6}{12.144} + \frac{36}{144.144} \text{ or}$$
$$1 + \frac{88}{144} + \frac{6.12}{12.12.144} + \frac{36}{144.144} \text{ or}$$
$$1 + \frac{88}{144} + \frac{72}{144.144} + \frac{36}{144.144} \text{ or}$$
$$1 + \frac{88}{144} + \frac{108}{144.144}$$

Note. The factors may be reduced to the lowest denomination, viz. lines, and then the product will be square lines, which divided by 144, will quote square

inches,

inches, and the remainder will be ſquare lines; and the ſquare inches divided by 144 will quote ſquare feet, and the remainder will be ſquare inches. Again, the ſquare feet divided by 9 will quote ſquare yards, and the remainder will be ſquare feet; and the ſquare yards divided by 36 will quote ſquare roods, and the remainder will be ſquare yards. Thus, becauſe in the foregoing example

the multiplicand is

$$38+\frac{9}{12}+\frac{6}{144}=\frac{38.144}{144}+\frac{9.12}{12.12}+\frac{6}{144}=\frac{5472}{144}+\frac{108}{144}$$
$$+\frac{6}{144}=\frac{5586}{144}.$$

and the multiplier

$$23+\frac{8}{12}+\frac{6}{144}=\frac{23.144}{144}+\frac{8.12}{12.12}+\frac{6}{144}=\frac{3312}{144}+\frac{96}{144}$$
$$+\frac{6}{144}=\frac{3414}{144}$$

the product will be

$$\frac{5586}{144}\times\frac{3414}{144}=\frac{19070604}{144\times 144}=\frac{132434}{144}+\frac{108}{144.144}$$
$$=919+\frac{98}{144}+\frac{108}{144.144}\text{ as before.}$$

RULE VI. *To multiply ſuperficial dimenſions by lineal dimenſions in order to have a product of a cubic dimenſion, write down the parts both of the multiplicand and multiplier with their reſpective denominators, as fractions of an integer; then work as in the preceding Rule IV. and laſtly, reduce thoſe parts of the product which are not reduced to the denomination of any cubic meaſure, either multiplying or dividing the fractions by which they are expreſſed, by a de-*

denominator fit for that purpose, as may be seen in the following

Example.

In a piece of timber, whose length is 18 feet 6 inches, breadth 2 feet 4 inches, and thickness 2 feet 3 inches, how many solid feet?

$$
\begin{array}{lll}
\textit{f.}\ \ \textit{in.} & & \\
18\ \ 6 = & 18 + \frac{6}{12} & \text{a lineal multiplicand.} \\
\ \ 2\ \ 4 = & 2 + \frac{4}{12} & \text{a lineal multiplier.} \\
 & \overline{}\times & \\
 & 37 & \text{superficial product by 2.} \\
\text{By Rule V.} & 6 + \frac{24}{144} & \text{superficial product by } \frac{4}{12}. \\
 & \overline{}+ & \\
 & 43 + \frac{24}{144} & \text{total square product.} \\
\ \ 2\ \ 3 = & 2 + \frac{3}{12} & \text{lineal multiplier.} \\
 & \overline{}\times & \\
 & 86 + \frac{48}{144} & \text{solid product by 2.} \\
 & 10 + \frac{9}{12} + \frac{72}{1728} & \text{solid product by } \frac{3}{12}. \\
 & \overline{}+ & \\
 & 96 + \frac{9}{12} + \frac{48}{144} + \frac{72}{1728} & \text{total solid prod. or} \\
 & 96 + \frac{9.144}{12.144} + \frac{48.12}{144.12} + \frac{72}{1728} & \text{or} \\
 & 96 + \frac{1296}{1728} + \frac{576}{1728} + \frac{72}{1728} & \text{or} \\
 & 96 + \frac{1944}{1728} = 97\frac{216}{1728} = 97 & \text{cubic feet 216 cubic inches.}
\end{array}
$$

Note. This operation may be facilitated by previously reducing the three factors to the lowest denomination, for instance, lines; which being multiplied continually, will produce cubic lines, which divided by 1728 will quote cubic inches, the remainder being cubic lines; and the cubic inches divided by 1728 will quote cubic feet, the remainder being cubic inches; and the cubic feet divided by 27 will quote cubic yards, the remainder being cubic feet; and the cubic yards divided

 by

by 216 will quote cubic roods, the remainder being cubic yards. Thus, in the preceding example,

$18\frac{6}{12}\times 2\frac{4}{12}\times 2\frac{3}{12}=\frac{222}{12}\times\frac{28}{12}\times\frac{27}{12}=\frac{167832}{1728}=97\frac{216}{1728}$ as before. Or, $\frac{222}{12}\times\frac{28}{12}\times\frac{27}{12}=\frac{111\cdot 2}{2\cdot 2\cdot 3}\times\frac{7\cdot 4}{3\cdot 4}\times\frac{3\cdot 3\cdot 3}{3\cdot 4}=\frac{111}{2}\times\frac{7}{4}=\frac{777}{8}=97\frac{1}{8}=97\frac{216}{1728}$ as before.

The reaſon of theſe operations depends upon the nature of fractions. See Sect. 3 of this Chapter.

SECTION IV.

DIVISION.

DEFINITION. *Diviſion* diſcovers how often one number is contained in another; or Diviſion, from two numbers given, finds a third, which contains unity as often as one number contains the other; or laſtly, Diviſion is the dividing any given number into any propoſed number of equal parts.

The number to be divided, or which contains the other, is called the *dividend*; the number by which we divide, or which is contained in the dividend, is called the *diviſor*; and the number found by diviſion, or which expreſſes how often the dividend contains the diviſor, is called the *quotient*, or *quot*.

As Multiplication ſupplies the place of many Additions, ſo Diviſion, which is the reverſe of Multiplication, ſerves inſtead of many Subtractions, as will thus appear: ſuppoſe it were required to divide 18 by 6, that is to find how often 6 is contained in 18, the work by Sub-

traction

traction will stand as in the margin; by which it appears that 6 is contained 3 times in the number 18. But this, by Division, may be found at one trial, thus:

$$\begin{array}{r} 18 \\ 6 \\ \hline 12 \\ 6 \\ \hline 6 \\ 6 \\ \hline 0 \end{array}$$

I set the divisor on the left of the dividend, leaving room on the right hand for the quotient, as in the margin; and then I say, How often 6 in 18? *Ans.* 3 times; this 3 I set in the quotient, then I multiply the quotient figure 3 into the divisor, saying, 3 times 6 make 18, which I set down below the dividend, and subtract it from the dividend, and nothing remains.

$$\begin{array}{l} \quad\quad\quad \text{Dividend.} \\ \text{Divisor } 6)18(3 \text{ Quotient.} \\ \quad\quad\quad\quad\, 18 \\ \quad\quad\quad\quad \overline{\quad 0} \end{array}$$

I. *Division of Integers.*

RULES. I. *From the left-hand part of the dividend point off the first dividual, viz. so many figures as will contain the divisor.*

II. *Ask how often the divisor is contained in the dividual, and put the answer in the quotient.*

III. *Multiply the divisor by the figure set in the quotient, and subtract the product from the dividual.*

IV. *To the right of the remainder bring down the next figure of the dividend for a new dividual, and then proceed as before.*

Notes.

Notes. I. The substance of these Rules is briefly expressed in the following verse:

Dic quot? multiplica, subduc, transferque sequentem.

First ask how oft? in quot the answer make;
Then multiply, subtract, and down a figure take.

II. Every remainder must be less than the divisor, for if it be either greater or equal, the divisor might have been oftener got, and the quotient figure is too little.

III. If any dividual happen to be less than the divisor, you must put 0 in the quotient, and bring down the next figure of the dividend, and if it be still less than the divisor, you must put another 0 in the quotient, and bring down the following figure of the dividend, &c.

IV. To complete the quotient, put the last remainder (if any) at the end of it above a small line with the divisor below it.

Example I.

Divisor.Dividend.Quotient.

```
 ...
7)875(125
  7
  --
  17
  14
  --
   35
   35
   --
    0
```

Here, because the divisor 7 is contained in 8, the left-hand figure of the dividend, I point it off as my first dividual, according to Rule I. and then I say: How often 7 in 8? Ans. once; which 1 I set in the quotient as directed in Rule II. then I multiply the divisor 7 by this quotient figure 1, and subtract the product 7 from the dividual 8, as directed in Rule III. to the remainder 1, I bring down the following figure 7 of the dividend and have 17 for my second dividual as directed in Rule IV. then I proceed as before, and say: How often 7 in 17? Ans. 2 times; where-

wherefore, ſetting 2 in the quotient, I multiply and ſubtract, and find the next remainder to be 3; to which I bring down the following figure 5 of the dividend and have 35 for my third dividual; then I ſay: How often 7 in 35? Anſ. 5 times, which 5 being placed in the quotient, I multiply and ſubtract, and 0 remains; ſo the quotient is 125.

Notes. I. Here obſerve, that the right-hand figure of the firſt dividual and all the ſubſequent figures of the dividend, have a point or dot ſet above them as they are brought down, which is done to prevent miſtakes by diſtinguiſhing them in this manner from the figures not yet brought down.

II. By reviewing the ſteps of the preceding operation and reducing, by Princ. VI. the dividuals and quotient-figures to their ſeparate values, the reaſon of the Rules will be obvious, as may be ſeen in the following operation:

$875=800+70+5$, and $125=100+20+5$.

Diviſor. Dividuals.	Partial Quots.
$7)800+70+5($	$100+20+5$
1ſt dividual 800	1ſt quot 100.
$7\times100=700$ −	ſubſtrahend.
remainder 100	
70 +	
2d dividual 170	2d quot 20.
$7\times20=140$ −	ſubtrahend.
remainder 30	
5 +	
3d dividual 35	3d quot 5.
$7\times5=35$ −	ſubtrahend.
0	total quot 125.'

Ex-

Example II.

$8)56032897(7004112\frac{1}{8}$

$7\times8=56$

..032 (Note III.)

$4\times8=32$

..8

$1\times8=8$

·9

$1\times8=8$

17

$2\times8=16$

1 (Note IV.)

Example III.

$36)789426(21928\frac{18}{36}$

$2.36=72$

. 69

$1.36=36$

334

$9.36=324$

. 102

$2.36=72$

. 306

$8.36=288$

18 (Note IV.)

Ex-

Example IV.

Let it be required to divide 170948*l.* among 234 men.

```
        234)170948(730 pounds.
   7.234=1638
         ----   -
          714
    3.234=702
          ---   -
  remainder 128      (Note III.)
             20
            ---x
        234)2560(10 shillings.
     1.234=234
           ---   -
  remainder 220      (Note III.)
             12
            ---x
            440
           220
           ---+
        234)2640(11 pence.
     1.234=234
           ---   -
           300
     1.234=234
           ---
  remainder 66
             4
            ---x
        234)264(1 30/234 farthings.
     1.234=234
           ---   -
            30       (Note IV.)
```

So each man's share is 730*l.* 10*s.* 11*d.* 1*f.* $\frac{30}{234}$.

Example V.

If 34168062 *cwt.* of goods be divided into 4875 equal lots, what will be the weight of each lot?

```
         4875)34168062(7008 cwt.
  7·4875=34125
         -----
            43062   (Note III.)
  8·4875=39000
         -----
remainder  4062
              4
           -----
    4875)16248(3 q.
 3·4875=14625
        -----
remainder  1623
             28
          -----
          12984
          3246
          -----
    4875)45444(9 1569/4875 lb.
 9·4875=43875
        -----
remainder  1569   (Note IV.)
```

So the weight of each lot is 7008 *cwt.* 3 *q.* 9 $\frac{1569}{4875}$ *lb.*

Note. In Examples IV. and V. instead of annexing the respective fractions to the integral parts of the quotients, each remainder is reduced to the next lower denomination, and the product thence arising by Multiplication is divided by the first given divisor.

Ex-

Example VI.

$48|00)9780|00(203\frac{36}{48}$

$2.48=96$

$\quad 180$ (Note III.)

$3.48=144$

$\quad 36$ (Note IV.)

Example VII.

$648|0)89678|2(138\frac{2542}{6480}$

$1.648=648$

$\quad 2487$

$3.648=1944$

$\quad 5438$

$8.648=5184$

$\quad 2542$ (Note IV.)

Note. The reaſon of cutting off ſome figures from the right hand of both the dividend and diviſor in theſe two laſt examples, will evidently appear by conſidering the following operations:

In Ex. VI. $\frac{978000}{4800}=\frac{9780.100}{48.100}=\frac{9780}{48}=203\frac{36}{48}$.

In Ex. VII. $\frac{896782}{6480}=\frac{896782}{648.10}=\frac{896782\div 10}{648}$

$=\frac{89678\frac{2}{10}}{648}=138\frac{254}{648}+\frac{2}{6480}=138+\frac{2542}{6480}$

Example VIII.

Divide 1692 by 468.

$$\frac{1692}{468}=\frac{1692\div 2}{468\div 2}=\frac{846}{234}=\frac{846\div 2}{234\div 2}=\frac{423}{117}=\frac{423\div 3}{117\div 3}=\frac{141}{39}$$
$$=\frac{141\div 3}{39\div 3}=\frac{47}{13}=3\frac{8}{13}.$$

Example IX.

Divide 7896 by 84.

$$\frac{7896}{84}=\frac{7896\div 2}{84\div 2}=\frac{3948}{42}=\frac{3948\div 2}{42\div 2}=\frac{1974}{21}=\frac{1974\div 3}{21\div 3}$$
$$=\frac{658}{7}=94.$$

Example X.

Divide 12.36 by 18.

$$\frac{12.36}{18}=12\times\frac{36}{18}=12\times 2=24.$$

Example XI.

Divide 8.72.48 by 4.24.72.

$$\frac{8.72.48}{4.24.72}=\frac{8.72.48}{4.72.24}=\frac{8}{4}\times\frac{72}{72}\times\frac{48}{24}=2.1.2=4.$$

Note. In these four examples the division is rendered more simple: but in computations or calculations that require the frequent use of the same divisor, the operation may be rendered more easy and expeditious, by making a table of the products of the divisor into all the nine digits, as in the following example; for then you have the quotient-figures and their products into the divisor by inspection.

Ex-

Example XII.

Divide 47896845 by 748.

TABLE.

1	748
2	1496
3	2244
4	2992
5	3740
6	4488
7	5236
8	5984
9	6732

748)47896845(64033$\frac{161}{748}$
4488
———
3016
2992
———
2484 (Note III.)
2244
———
2405
2244
———
161 (Note IV.)

II. *Division of Decimals.*

RULE. *Divide as in Integers, pointing off as many Decimals in the quotient, as the dividend has more than the divisor, and whenever the number of figures in the quotient is less than the required number of Decimals, prefix cyphers to supply the defect, as in Ex. IV.*

Notes. I. The reason of this Rule will easily appear by considering, that the dividend becomes ten times, hundred times, &c. greater, as it is divided by tenths, by hundredths, &c.

II. If the decimals of the divisor exceed those of the dividend, you must add to the dividend so many cyphers as are required to make the decimals of the dividend equal or surpass those of the divisor, as in Ex. III.

III. If

III. If in any caſe there be a remainder after all the dividend figures are uſed, the quotient may be continued to what number of decimals you pleaſe by ſubjoining a cypher continually to the laſt remainder, as in Examples V. and VI.

Example I.

```
        ...
  .75).58875(.785
7×75=525
      ----
      .637
 8×75=600
      ----
       .375
  5×75=375
       ----
        ...
```

Example II.

```
        .
  2.5)182.5(73
7×25=175
     ----
      ..75
  3×25=75
      ---
       ..
```

Example III.

```
  .375)12.75(
              ..
  .375)12.750(34
3×375=1125
      ----
      .1500
 4×375=1500
      -----
       ....
```

Example IV.

```
          ...
  2.5).22875(.0915
9×25=225
     ---
     ..37
 1×25=25
      ---
      125
 5×25=125
      ---
      ...
```

Ex-

Example V.

.8)29(36.25
3×8=24
50
6×8=48
.20
2×8=16
.40
5×8=40

Example VI.

.018).0024(.133, &c.
1×18=18
.60
3×18=54
.60
3×18=54
6, &c.

III. *Division of Fractions.*

RULE. *Multiply cross-ways, viz. the numerator of the dividend into the denominator of the divisor for the numerator of the quot, and the denominator of the dividend into the numerator of the divisor for the denominator of the quot.*

Or, Invert the divisor, and then multiply it into the dividend.

Note. The reason of the Rule will appear by considering that the method here used is nothing else but the reducing the divisor and dividend to a common denominator, and then dividing one numerator by the other. Thus $\frac{2}{3} \div \frac{3}{4} = \frac{2\cdot4}{3\cdot4} \div \frac{3\cdot3}{4\cdot3} = \frac{8}{12} \div \frac{9}{12} = \frac{8}{9}$.

Examples.

Examples.

I. $\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2} = \frac{4 \cdot 3}{5 \cdot 2} = \frac{12}{10} = 1.2 = 1\frac{2}{10} = 1\frac{1}{5}$.

II. $4\frac{1}{4} \div \frac{3}{4} = 4\frac{1}{4} \times \frac{4}{3} = \overline{\frac{4}{1} + \frac{1}{4}} \times \frac{4}{3} = \frac{16+1}{4} \times \frac{4}{3} = \frac{17}{4} \times \frac{4}{3}$
$= \frac{17}{3} = 5\frac{2}{3}$.

III. $\frac{7}{8} \div 4\frac{2}{3} = \frac{7}{8} \div \overline{\frac{4}{1} + \frac{2}{3}} = \frac{7}{8} \div \frac{12+2}{3} = \frac{7}{8} \div \frac{14}{3} = \frac{7}{8} \times \frac{3}{14} =$
$\frac{7}{14} \times \frac{3}{8} = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$.

IV. $6\frac{5}{8} \div 3\frac{1}{4} = \overline{\frac{6}{1} + \frac{5}{8}} \div \overline{\frac{3}{1} + \frac{1}{4}} = \frac{53}{8} \div \frac{13}{4} = \frac{53}{8} \times \frac{4}{13} =$
$\frac{53}{13} \times \frac{4}{8} = \frac{53}{13} \times \frac{1}{2} = \frac{53}{26} = 2\frac{1}{26}$.

V. $\frac{3}{4} \times \frac{5}{6} \div \frac{7}{8} = \frac{3}{4} \times \frac{5}{6} \times \frac{8}{7} = \frac{3}{4} \times \frac{5}{2 \times 3} \times \frac{8}{7} = \frac{3}{8} \times \frac{5}{3} \times \frac{8}{7} = \frac{5}{7}$.

VI. $\frac{4}{5} \times \frac{5}{6} \div \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{4}{5} \times \frac{5}{6} \times \frac{4}{3} \times \frac{3}{2} \times \frac{2}{1} = \frac{4 \times 4}{6} = \frac{2 \times 4}{3} = \frac{8}{3}$
$= 2\frac{2}{3}$.

IV. *Division of several Denominations.*

RULE I. *If the divisor be an integer, divide the integers of the dividend by it, reduce the remainder to the parts of the next inferior denomination and add it, when thus reduced to the said parts; then divide the sum, reducing and adding the remainder to the parts of the following denomination, &c.*

Note. If the integral part of the dividend be less than the divisor you must in the first place reduce it to the parts of the next denomination, as in Ex. II.

Ex-

Example I.

If 274*l.* 13*s.* 8*d.* be equally divided among 8 men, what will each man's share be?

```
Divisor.   Dividend.      Quotient.
            l.  s.  d.    l.  s.  d. f.
     8 )  274  13   8  (  34   6   8  2
           24
           --  -
           34
           32
           --  -
remainder . 2 pounds to be reduced into shillings.
           20
           --  x
           40 shillings.
           13
           --  +
     8 )   53 ( 6 shillings.
           48
           --  -
rem. .      5 shill. to be reduced into pence.
           12
           --  x
           60 pence.
            8
           --  +
     8 )   68 ( 8 pence.
           64
           --  -
rem. .      4 pence to be reduced into farthings.
            4
           --  x
     8 )   16 ( 2 farthings.
           16
           --  -
            0
```

Example II.

If 1*cwt.* or 112*lb.* of nutmegs be valued at 76*l.* 18*s.* 8*d.* what is that per lb.?

```
Divisor.   Dividend.     Quotient.
           l.  s.  d.    l.  s.  d.  f.
  112 )    76  18   8 (  0  13   8   3 48/112
           20
          ----  ×
          1520
            18
          ----  +
  112 )   1538 ( 13 shillings.
          112
          ----  -
          .418
           336
          ----  -
   rem.   . 82 shill. to be reduced into pence.
            12
          ----  ×
           984 pence.
             8
          ----  +
  112 )    992 ( 8 pence.
           896
          ----  -
   rem.   . 96 pence to be reduced into farthings.
             4
          ----  ×
  112 )    384 ( 3 farthings.
           336
          ----  -
   rem.   . 48
```

(See Note IV. p. 54.)

RULE

RULE II. *If the divisor be a fraction, multiply its denominator into all the parts of the dividend, and then divide the product by the numerator of the same fraction according to the foregoing Rule.*

Example.

What is the price of a yard of velvet, $\frac{2}{3}$ of which are valued at 2*l.* 5*s.* 6*d.* 2*f.*$\frac{2}{3}$?

Divisor.	Dividend.				Quotient.		
	l.	*s.*	*d.*	*f.*	*l.*	*s.*	*d.*
$\frac{2}{3}$)	2	5	6	$2\frac{2}{3}$	(3	8	4
					$\times 3$		
	6	16	8	0	product by denom. 3.		
					$\div 2$		
	3	8	4	0	quotient by num. 2.		

RULE III. *If the divisor consists of integers and parts, reduce both divisor and dividend to the same denomination, and then proceed as in Division of Integers.*

Example I.

A nobleman distributes among some poor people 20*l.* 5*s.* each person got 7*s.* 6*d.* what was the number of the poor?

s.	*d.*	*l.*	*s.*	Divisor.	Dividend.	Quotient.
7	6	20	5	9|0)	486|0 (	54 persons.
12		20			45	
	×		×			
90	pence.	405	shill.		36	
		12			36	
			×			
		4860	pence.		0	

Example II.

The content of a rectangular floor or pavement is 452 square feet and 107 square inches; the breadth is 18 feet and 5 inches; what is the length?

Lineal Divisor.		Square Dividend.		Quotient.
f. in.		f. in.		f. in.
18 5	)	452 107	(	24 7
—— × 12		—— × 12		
216		5424		
5		—— × 12		
—— +		65088		
221		107		
—— × 12		—— +		
2652	)	65195	(	24 lineal feet.
		5304		
		—— −		
		12155		
		10608		
		—— −		
	rem.	1547 lineal feet.		
		—— × 12		
2652	)	18564	(	7 lineal inches.
		18564		
		—— −		
		0		

Notes. I. This last example may be more easily performed by reducing the dividend to square inches, and the divisor to lineal inches, and then dividing, the quotient will be lineal inches. Thus,

				Divisor.	Dividend.	Quotient.
f.	*in.*	*f.*	*in.*	*lin. in.*	*sq. in.*	*lin. in.*
18	5	452	107	221)	65195 (	295
12		144			442	÷12
×		×				24*f.* 7*in.*
221		65195			2099	
					1989	
					1105	
					1105	
					0	

II. *But in general you may reduce the divisor to the lowest denomination, then divide according to Rule II.* Thus, because 7*s.* 6*d.* make 90*d.* or $\frac{90}{240}$ of 1*l.* that is $\frac{3}{8}$ of 1*l.* and 18*f.* 5*in.* make 221 *inches*, or $\frac{221}{12}$ of 1*f.* it will be

In the 1st Example.				In the 2d Example.			
	l.	*s.*			*f.*	*in.*	
Divis. $\frac{3}{8}$)	20	5	Divid.	Divis. $\frac{221}{12}$)	452	107	Divid.
			×8				×12
	162	0			5432	11	
			÷3				÷221
	54	0	quotient.		24	7	quot.

SECTION V.

THE PROOFS OF THESE OPERATIONS.

I. *The Proof of Addition.*

RULE I. *By Addition: add each column first upwards and then downwards, and if you find the sum to be the same both ways, conclude the work to be right.*

Ex-

Example.	Addition upwards.	Addition downwards.
836	6+4+5=..15	5+4+6=..15
474	3+7+0=.10.	0+7+3=.10.
905	8+4+9=21..	9+4+8=21..
—— +	—— +	—— +
2215	2215 Total.	2215

RULE II. *By Subtraction: subtract one of the numbers added from the sum, the remainder will be equal to all the others together.*

Example I.		Example II.	
836	1st number.	836	1st number.
474	2d number.	474	2d number.
——	+	905	3d number.
1310	total.	——	+
474	2d number.	2215	total.
——	−	905	3d number.
836	1st number.	——	−
		1310	sum of 1st and 2d.

II. *The Proof of Subtraction.*

RULE I. *By Addition: add the remainder to the minor, the sum will be equal to the major.* See Princ. VIII,

Example I.		Example II. *l.*	*s.*	*d.*	
5847	major.	73	15	10	major.
2569	minor.	48	12	6	minor.
——	−	——			−
3278	remainder.	25	3	4	remainder.
2569	minor.	48	12	6	minor.
——	+	——			+
5847	major.	73	15	10	major.

RULE

RULE II. *By Subtraction: subtract the remainder from the major, the difference will be equal to the minor.* See Principle VIII.

Ex. I.

5847	major.
2569	minor.
3278	remainder.
5847	major.
2569	minor.

Ex. II.

l.	*s.*	*d.*	
73	15	10	major.
48	12	6	minor.
25	3	4	remainder.
73	15	10	major.
48	12	6	minor.

III. *The Proof of Multiplication.*

RULES I. *By Multiplication: change the places of the factors, and make that the multiplier which before was the multiplicand; and, if the work be right, you will have the same product as before.*

II. *By Division: When the work is right, the product divided by the multiplier quotes the multiplicand, or divided by the multiplicand quotes the multiplier.*

IV. *The Proof of Division.*

RULES. I. *By Multiplication: multiply the quotient by the divisor, or the divisor by the quotient, and the product with the remainder (if any) added to it, will be equal to the dividend.*

II. *By Division: divide the difference of the dividend and remainder by the quotient, and your next quotient will be equal to your first divisor, without any remainder.*

SEC-

SECTION VI.

COMPOUND OPERATIONS.

DEFINITIONS. I. An operation is ſaid to be *compound* when it is performed upon the numbers connected by the ſigns + or −. See the Introduction.

II. Simple numbers, or the terms of compound numbers, are ſaid to be *like* when they ſtand together or in the ſame place, that is, one under another; but they are *unlike* if they ſtand alone. Thus the numbers −2 and 12, which ſtand together, or one under another, as in the margin, are like numbers, but 8 and −3, which ſtand alone, are unlike.

$$\begin{array}{rl} 8 & -\ 2 \\ 12 & -\ 3 \end{array}$$

I. *Compound Addition.*

RULES. I. *Add together terms that are like and have like ſigns, then to their ſum prefix the common ſign.*

II. *To add terms that are like but have unlike ſigns, ſubtract the leſs from the greater, and prefix the ſign of the greater to the remainder.*

III. *To add terms that are unlike, ſet them all down with their ſigns in their proper place.*

IV. *Compound numbers are added together by uniting the ſeveral terms of which they conſiſt, by the preceding Rules.*

Examples.

Examples.

$$\text{I.}\quad \begin{array}{r} 3+6-4+5+7 \\ 5-2-3-12+4 \\ \hline \text{Sum } 3+11-6+2-5+4 \end{array} +$$

$$\text{II.}\quad \begin{array}{r} 10-4+8-9 \\ -6-3+4+2 \\ 1+5-6-8 \\ \hline \text{Sum } 10-10+6+0-4-8 \end{array} +$$

II. *Compound Subraction.*

RULE. *Change the signs of the terms to be subtracted into the contrary signs, and then add them, so changed, to the compound number from which they were to be subtracted, by Compound Addition; the sum hence arising is the remainder.*

Example.

$$\begin{array}{r} 5-7+4-5+8+2 \\ 2-4-3+5+2-6 \\ \hline \text{Diff. } 5-9+8-2+3+0+6 \end{array} -$$

$$\begin{array}{r} \text{Or, } 5-7+4-5+2+2 \\ -2+4+3-5-2+6 \\ \hline \text{Sum } 5-9+8-2-3+0+6 \end{array} +$$

Note. The truth of the Rule may by proved thus: $6-4=2$, but $4=6-2$; then $6-4=6-(6-2)=6-6+2=2$, as before; otherwise the remainder would not be right.

I. III.

III. *Compound Multiplication.*

RULES. I. *When the ſigns of the two terms to be multiplied are like, the ſign of the product is* $+$, *but when the ſigns are unlike, the ſign of the product is* $-$.

II. *To multiply two terms, find the ſign of the product by Rule I. and after it place the product of the numbers.*

III. *To multiply compound numbers, beginning at the right, in order to write more eaſily, multiply every term of the multiplier by all the terms of the multiplicand, one after anoth r, according to the preceding rules; place the firſt term of each particular compound product under the multiplying term, and then collect all theſe products into one ſum, that ſum is the product required.*

Note. The reaſon of the firſt Rule may be ſhewn thus: becauſe $6 \times 4 = 24$, and $4 = 6 - 2$, it will be $6 \times 4 = 6 \times \overline{6 - 2} = 6 \times 6 - 2 \times 6 = 36 - 12 = 24$. Again, $6 = 7 - 1$; and therefore, $6 \times 4 = \overline{7 - 1} \times \overline{6 - 2} = \overline{7 - 1} \times 6 + \overline{7 - 1} \times -2 = 7 \times 6 - 1 \times 6 + 7 \times -2 - 1 \times -2 = 42 - 6 - 14 + 2 = 44 - 20 = 24$.

Example.

$$
\begin{array}{rrrrrl}
 & 4 & -3 & +2 & & \\
 & 3 & -1 & -6 & & \\
\hline
 & & & & & \times \\
 & 12 & -9 & +6 & & \\
 & & -4 & +3 & -2 & \\
 & & & -24 & +18 & -12 \\
\hline
 & & & & & + \\
\text{Product} & 12 & -13 & -15 & +16 & -12
\end{array}
$$

IV. *Compound Division.*

RULES. I. *If the signs of the two terms to be divided are like, the sign of the quotient is +, if they are unlike, the sign of the quotient is −.*

II. *The first term of the dividend is to be divided by the first term of the divisor, observing the preceding Rule for the signs; and this quotient being set down as a part of the quotient wanted, is to be multiplied by the whole divisor, and the product subtracted from the dividend; if nothing remain, the division is finished; the remainder, when there is any, is a new dividend.*

III. *Divide the first term of this new dividend by the first term of the divisor as before, and join the quotient to the part already found, with its proper sign; then multiply the divisor by this part of the quotient, and subtract the product from the new dividend; and thus the operation is to be continued till no remainder is left, or till it appear that there will always be a remainder.*

Note. The first Rule is easily deduced from Rule I. of Compound Multiplication; for being, for instance, $-4 \times 3 = -12$, it will also be $-4 = \frac{-12}{3}$, and $3 = \frac{-12}{-4}$. See Rule II. of the Proof of Multiplication.

Example I.

$$
\begin{array}{l}
4-3+2)12-13-15+16-12(3-1-6.\ \text{Quotient.} \\
12-\ 9+\ 6 \\
\hline
-\ 4-21+16-12 \\
-\ 4+\ 3-\ 2 \\
\hline
-24+18-12 \\
-24+18-12 \\
\hline
(0)
\end{array}
$$

Example II.

$$
\begin{array}{l}
1-2)1\quad(1+2+4+8+16+\ \&c.\ \text{Quotient.} \\
1-2 \\
\hline
2 \\
2-4 \\
\hline
4 \\
4-8 \\
\hline
8 \\
8-16 \\
\hline
16 \\
16-32 \\
\hline
32,\ \&c.
\end{array}
$$

Ex.

Example III.

$$
\begin{array}{l}
1-2+1)1 \qquad (1+2+3+4+5+\ \&c.\ \text{Quotient.} \\
\quad 1-2+1 \\
\hline
\qquad 2-1 \\
\qquad 2-4+2 \\
\hline
\qquad\quad 3-2 \\
\qquad\quad 3-6+3 \\
\hline
\qquad\qquad 4-3 \\
\qquad\qquad 4-8+4 \\
\hline
\qquad\qquad\quad 5-4 \\
\qquad\qquad\quad 5-10+5 \\
\hline
\qquad\qquad\qquad 6-5,\ \&c.
\end{array}
$$

Note. It often happens, as in the two last examples, that there is still a remainder from which the operation may be continued without end. This expression of a quotient is called *an infinite series*; the nature of which shall be considered afterwards. By comparing a few of the first terms the law of the series may be discovered, by which, without any more division, it may be continued to any number of terms wanted. Thus, because the terms of the quotient in Ex. II. increase by the same multiplier 2, they form a geometrical series; but the terms of the quotient in Ex. III. increasing by the common difference 1, they form an arithmetical series, which is the natural scale of numbers, as is obvious.

SECTION VII.

RESOLUTION,

WHEREIN OF THE DIVISORS OF NUMBERS.

I. *Resolution.*

DEFINITION. *Resolution* is the finding of the component parts of a composite number, for instance, of 30, whose component parts are 1, 2, 3, and 5, because $1 \times 2 \times 3 \times 5 = 30$.

RULE. *Divide the given sum by its less divisor, then the quotient by its less divisor, and so continually divide the new quotient by its less divisor till the last quotient be* 1; *all these single divisors will be the component parts of the given number.*

Example.

		1	Simple divisors or component parts.
Composite number	360	2	
1st quotient	180	2	
2d quotient	90	2	
3d quotient	45	3	
4th quotient	15	3	
5th quotient	5	5	
6th and last quotient	1		

De-

Demonſtration.

1	1ſt diviſor.
2	2d diviſor.
—	×
2	
2	3d diviſor.
—	×
4	
2	4th diviſor.
—	×
8	
3	5th diviſor.
—	×
24	
3	6th diviſor.
—	×
72	
5	7th diviſor.
—	×
360	the given number.

Note. Unity is ſet down among the diviſors as a component part of every number.

II. *Of the Diviſors of Numbers.*

RULE I. *To find all the diviſors of a given number, begin with finding its ſimple diviſors, then multiply the 3d ſimple diviſor by the 2d, the firſt being unity, and write down the product, which will be a compoſite diviſor of the given number; again, multiply the 4th ſimple diviſor ſeparately by the 2d and 3d, and alſo by the compoſite diviſor, in order to get three new compoſite diviſors; and ſo continually multiply the next following ſimple diviſor by every foregoing either ſimple or compoſite diviſor, till you have a product equal to the given number.*

Note.

Note. If in the courſe of the operation you obtain a product equal to any of the preceding compoſite diviſors, there is no occaſion to write it down.

Example.

To find all the diviſors of 360.

	ſimple diviſors.	Compoſite Diviſors.
	1	
360	2	
180	2	4
90	2	8
45	3	6, 12, 24
15	3	9, 18, 36, 72
5	5	10, 15, 20, 30, 40, 45, 60, 90, 120, 180, 360.
1		

Anſwer 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360.

RULE II. *To find the greateſt common diviſor of two given numbers, divide the greater by the leſs, and again divide the diviſor by the remainder, and ſo on continually, till* 0 *remains. The laſt diviſor is their greateſt common diviſor.*

Example.

To find the greateſt common diviſor of the numbers 784 and 952.

```
                        784)952(1
                            784
                            ---
                            168)784(4
                                672
                                ---
                                112)168(1
                                    112
                                    ---
Greateſt common diviſor              56)112(2
                                        112
                                        ---
                                          0
```

Notes.

Notes. I. The reason of the Rule is, because the last remainder dividing exactly the next preceding, it will also divide its multiples, and therefore the dividend itself. This will still farther appear by resolving the given numbers, as follows:

$112=2\times56.$
$168=112+56=2\times56+56=3\times56.$
$784=4\times168+112=12\times56+2\times56=14\times56.$
$952=784+168=14\times56+3\times56=17\times56.$

II. If you find all the divisors of the two given numbers, you will find likewise among these the greatest common divisor. Thus, the divisors of 952 being 1, 2, 4, 7, 8, 14, 17, 28, 34, 56, 68, 119, 136, 238, 476, 952; and those of 784 being 1, 2, 4, 7, 8, 14, 16, 28, 49, 56, 98, 112, 196, 392, 784, it is obvious, that 56 is the greatest common divisor.

SECTION VIII.

REDUCTION.

DEFINITIONS. I. *Reduction* teacheth how to bring a number of one name or denomination to another of the same value, and is either *Descending*, or *Ascending*, or *Mixed*.

II. *Reduction Descending* brings a number of a higher denomination to a lower, when the lower is some aliquot part of the higher, as pounds to shillings, pence, or farthings, and is performed by Multiplication.

III. *Reduction Ascending* brings a number of a lower denomination to a higher, when the lower is some aliquot part of the higher, as shillings, pence, or farthings, to pounds, and is performed by Division.

 IV. *Mixed*

IV. *Mixed Reduction* brings a number of one denomination to another, when the one is no aliquot part of the other, as pounds to guineas, and requires the use of both Multiplication and Division.

RULE. *Multiply or divide as the Tables of coin, weights, and measures, direct.*

I. *Reduction Descending, or Ascending.*

Quest. I. In 472*l.* how many shillings, pence, and farthings?

Answer by Reduction Descending.

```
   472 pounds.
    20 shill. in 1 pound.
  ----- ×
  9440 shillings.
    12 pence in 1 shill.
------ ×
113280 pence.
     4 farth. in 1 penny.
------ ×
453120 farthings.
```

Proof by Reduction Ascending.

```
 4 ) 453120 farthings.
     ------
12 ) 113280 pence.
     ------
2|0 ) 944|0 shillings.
      -----
        472 pounds.
```

Quest. II. In 458*l.* 16*s.* 7*d.* $\frac{3}{4}$ how many shillings, pence, and farthings?

Answer by Reduction Descending.

```
           458l. 16s. 7d. 3f.
              20 I take in the 16 shill.
           ----- ×
shillings   9176
              12 I take in the 7 pence.
          ------ ×
pence     110119
               4 I take in the 3 farthings.
          ------ ×
farthings 440479
```

Proof

Proof by Reduction Ascending.

```
                      Remainders.
    4 ) 440479f.        3f.
        -------
   12 ) 110119d.        7d.
        -------
   2|0 ) 917|6s.        16s.
         -------
         458l. 16s. 7d. 3f.
```

II. *Mixed Reduction.*

RULE. *By Reduction Descending bring the given name to some such third name as is an aliquot part both of the name given and of the name sought, and then by Reduction Ascending bring the third name to the name sought.*

Quest. 1. In 764*l.* how many guineas?

Operation.

```
         764 pounds.
          20
        ------ ×
  21 ) 15280s. ( 727 guineas.
       147
       ----- -
       ..58
         42
         --- -
         160
         147
         --- -
remainder 13 shillings.
```

 Proof.

Proof.

In 727 guineas 13 shillings, how many pounds?

```
Guineas. Shillings.
      727   13
       21
      ----  ×
      727
     1454
       13
     -----  +
2|0 ) 1528|0 shillings.
     -----
      764 pounds.
```

Quest. II. In 4785*l*, 13*s*, how many pieces of 13½*d*. per piece?

Operation.

```
13½            4785 13
 2               20
--  ×          ------  ×
27 half-pence.  95713 shillings.
                   24 half-pence in 1 shill.
               ------  ×
               382852
              191426
              -------  +
      27 ) 2297112 half-pence. ( 85078 pieces
           216                       sought.
           ---  -
            137
            135
            ---  -
              211
              189
              ---  -
               222
               216
               ---  -
    remainder 6 half-pence or 3d.
```

Proof.

Proof.

In 85078 pieces of 13½*d.* each, and 3*d.* how many pounds?

Pieces. Pence.

85078 $3 = \frac{6}{2}$

27 half-pence in 1 piece.

×

595552 I take in the 6 half-pence.

170150

\+

24) 2297112 half-pence. (9571|3 shill.

216 $\div$ 2|0

— 4785*l.* 13*s.*

137

120

—

171

168

—

31

24

—

72

72

—

0

Note. The 2297112 half-pence are divided by 27 in the operation, and by 24 in the proof, because 27 half-pence make a piece of 13½*d.* and 24 half-pence make a shilling.

Quest. III. In 872*l.* how many pieces of 6*d.* of 5*d.* and of 4*d.* of each an equal number?

Operation.

```
d.            l.
6            872
5              240 pence in 1 pound.
4            ----- ×
— +          34889
15 pence.   1744
            ------ +

15 ) 209280d. ( 13952 pieces of 6d. of
     15                   5d. and 4d.
     -- -
      59
      45
      -- -
      142
      135
      --- -
        78
        75
        -- -
         30
         30
         -- -
          0
```

Note. The 209280 pence are divided by 15, because 15 pence make up the value of a piece of 6*d.* another of 5*d.* and a third of 4*d.*

In

Proof.

In 13952 pieces of 6*d.* of 5*d.* of 4*d.* of each an equal number, how many pounds?

$$\begin{array}{rrl}
 & 13952 & \text{pieces.} \\
 & 15 & \text{pence in 1 piece.} \\
\hline
 & 69760 & \\
 & 13952 & \\
\hline
24|0\ ) & 20928|0d. & (\ 872l. \\
 & 192 & \\
\hline
 & 172 & \\
 & 168 & \\
\hline
 & 48 & \\
 & 48 & \\
\hline
 & 0 &
\end{array}$$

III. *Reduction of Fractions.*

WE subjoin here this operation as depending upon the same principles.

Probl. I. To reduce an improper fraction, $\frac{595}{7}$, $\frac{437}{8}$, &c. to an integer, or mixed number.

Work by Division, and you will find $\frac{595}{7} = 85$ and $\frac{437}{8} = 54\frac{5}{8}$.

Probl.

Probl. II. To reduce a whole number to a fraction of a given denominator.

Let 9 be the number and 5 the denominator, it will be $9=\frac{9\times5}{5}=\frac{45}{5}$, the fraction required.

Probl. III. To reduce a mixed number to an improper fraction.

Suppose $54\frac{5}{8}$ the mixed number, then $54\frac{5}{8}=\frac{54\times8}{8}+\frac{5}{8}=\frac{432}{8}+\frac{5}{8}=\frac{437}{8}$ will be an improper fraction.

Probl. IV. To reduce a compound fraction, as $\frac{2}{3}$ of $\frac{4}{5}$, or $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$, to a simple one.

Work by Multiplication; thus, $\frac{2}{3}$ of $\frac{4}{5}=\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$, and $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$.

Probl. V. To reduce a fraction of a greater denomination to a fraction of a less denomination.

What fraction of a shilling is $\frac{3}{4}$ of a pound?

Here, as in Reduction Descending, $\frac{3}{4}l.=\frac{3}{4}\times20s.=\frac{60}{4}s.=15s.=\frac{15}{20}$.

Probl. VI. To find the value of a fraction.

What is the value of $\frac{3}{4}$ of a pound?

$\frac{3}{4}l.=\frac{3}{4}\times20s.=\frac{60}{4}s.=15s.=\frac{15}{20}$.

What is the value of $\frac{9}{16}$ of a pound?

$\frac{9}{16}l.=\frac{9}{16}\times20s.=\frac{180}{16}s.=11\frac{4}{16}s.=11s.+\frac{4}{16}\times12d.=11s.+3d.$

Probl.

Probl. VII. To reduce a fraction to its lowest terms.

Reduce $\frac{784}{952}$ to its lowest terms.

$$\frac{784}{952}=\frac{784\div2}{952\div2}=\frac{392\div2}{476\div2}=\frac{196\div2}{238\div2}=\frac{98\div7}{119\div7}=\frac{14}{17}$$

fraction required.

Or, $\frac{784}{952}=\frac{784\div56}{952\div56}=\frac{14}{17}$ as before.

Note. The number 56 is the greatest common divisor of the numerator and denominator of the given fraction. See Section VII. Rule II. where of the divisors of numbers.

Probl. VIII. To reduce fractions of different denominators to a common denominator.

See the Lemma in the Addition of Fractions, Sect. I. of this Chapter.

LEMMA. To find the least multiple of two or more numbers.

RULE. *Place the given numbers in a line; then divide two of them, or as many of them as you can, by 2 or 3, or any small divisor that leaves no remainder, place the quotients and the numbers not divided in a line below; again, divide the numbers in this line either by the former or by some other divisor, placing the quotients and the numbers not divided in a line below; proceed by dividing in the same manner till the last quotient of every number be unity; then multiply the divisors into one another continually, and their product is the least multiple required.*

Example I.

Required the least multiple of 24 and 36?

Divisors	
6	24, 36
2	4, 6
2	2, 3
3	1, 3
	1

The least multiple is $6 \times 2 \times 2 \times 3 = 72$.

Example II.

Required the least multiple of 24, 36, 54.

Divisors	
6	24, 36, 54
3	4, 6, 9
2	4, 2, 3
2	2, 1, 3
3	1, 3
	1

The least multiple is $6 \times 3 \times 2 \times 2 \times 3 = 216$.

Probl. IX. To reduce fractions to the lowest or least common denominator possible.

RULE. *Find the least multiple of all the given denominators by the preceding Lemma, and it will be the least common denominator; then divide this common denominator severally by the denominators of the given fractions, and multiply each numerator respectively by the quotient arising from its own denominator, and the products will be the numerators.*

Ex. I. Reduce $\frac{5}{12}$ and $\frac{7}{18}$ to their least common denominator.

Divisors	
6	12, 18
2	2, 3
3	1, 3
	1

6 x 2

$6 \times 2 \times 3 = 36$ the least common denominator.

$5 \times 36 \div 12 = 5 \times 3 = 15$ the first numerator.
$7 \times 36 \div 18 = 7 \times 2 = 14$ the second numerator.

The new fractions are $\frac{15}{36}$ and $\frac{14}{36}$.

Ex. II. Reduce $\frac{5}{8}$, $\frac{7}{12}$, $\frac{4}{9}$, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{1}{4}$ to their least common denominator.

Divisors	2	8,	12,	9,	3,	6,	4
	2	4,	6,	9,	3,	3,	2
	2	2,	3,	9,	3,	3,	1
	3	1,	3,	9,	3,	3	
	3		1,	3,	1,	1	
				1			

$2 \times 2 \times 2 \times 3 \times 3 = 72$ the least common denominator.

$5 \times 72 \div 8 = 5 \times 9 = 45$, first numerator.
$7 \times 72 \div 12 = 7 \times 6 = 42$, second numerator.
$4 \times 72 \div 9 = 4 \times 8 = 32$, third numerator.
$2 \times 72 \div 3 = 2 \times 24 = 48$, fourth numerator.
$5 \times 72 \div 6 = 5 \times 12 = 60$, fifth numerator.
$1 \times 72 \div 4 = 1 \times 18 = 18$, sixth numerator.

New fractions $\frac{45}{72}$, $\frac{42}{72}$, $\frac{32}{72}$, $\frac{48}{72}$, $\frac{60}{72}$, $\frac{18}{72}$.

Probl. X. To reduce a vulgar fraction to a decimal.

Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{16}$, $\frac{1}{3}$, $\frac{4}{15}$, $\frac{1}{11}$, to decimals.

I. $\frac{1}{2}=\frac{1.10}{2.10}=\frac{10\div 2}{10}=\frac{5}{10}=.5$

II. $\frac{3}{4}=\frac{3}{4}\times\frac{100}{100}=\frac{300\div 4}{100}=\frac{75}{100}=.75$

III. $\frac{1}{16}=\frac{1}{16}\times\frac{10000}{10000}=\frac{10000\div 16}{10000}=\frac{625}{10000}=.0625$

IV. $\frac{1}{3}=\frac{1}{3}\times\frac{100}{100}=\frac{100\div 3}{100}=\frac{33\frac{1}{3}}{100}=.33$, &c.

V. $\frac{4}{15}=\frac{4}{15}\times\frac{1000}{1000}=\frac{4000\div 15}{1000}=\frac{266\frac{10}{15}}{1000}=.266$, &c.

VI. $\frac{1}{11}=\frac{1}{11}\times\frac{10000}{10000}=\frac{10000\div 11}{10000}=\frac{909\frac{1}{11}}{10000}=.09,09,$ &c.

Notes. I. In reducing a vulgar fraction to a decimal, if 0 at last remains, as in the three first Examples, the decimal is precisely equal to the vulgar fraction, and is called a *finite* or *terminate decimal*: but, if there is any remainder, as in the three last Examples, the decimal thence resulting cannot be precisely equal to the vulgar fraction, and is called *infinite* or *interminate*; of which there are two sorts; for some constantly repeat the same figure, as in Ex. IV. and V. and are called *repeating decimals*, *repeaters*, or *single repetends*; others repeat a circle of figures, as in Example VI. and on that account are called *circulating decimals*, *circulates*, or *compound repetends*.

II. This problem serves to reduce the parts of coin, weight, measure, time, &c. to decimals. Thus $9d.=\frac{9}{12}s.$ $=\frac{9}{12}\times\frac{100}{100}=\frac{75}{100}=.75$ of a shilling; or, $9d.=\frac{9}{240}$ of a pound $=\frac{9}{240}\times\frac{10000}{10000}=\frac{90000\div 240}{10000}=\frac{375}{10000}=$ .0375 of a pound.

Probl.

Probl. XI. To reduce a decimal to value.

Reduce .875*l.* to value.

$.875 \times 20 = 17.500s. = 17s. + .5 \times 12d. = 17s. + 6.0d.$
$= 17s.\ 6d.$

Probl. XII. To reduce a decimal to its primitive vulgar fraction.

RULE I. *When the given decimal is finite, divide both the numerator and the denominator by their greatest common measure, the quotient is the vulgar fraction required.* Thus

$.875 = \frac{875}{1000} = \frac{875 \div 125}{1000 \div 125} = \frac{7}{8}$, and $.0625 = \frac{625}{10000}$
$= \frac{625 \div 625}{10000 \div 625} = \frac{1}{16}$.

RULE II. *When the given decimal is a pure repeater or a pure circulate, that is, without any finite part, make the repeating figure, or the figures of the circle, the numerator of the vulgar fraction; the denominator is 9 for the repeating figure, or 9 for every figure of the circle; and then, if occasion requires, reduce this fraction to its lowest terms.*

Thus, .333, &c. $= \frac{3}{9} = \frac{3 \div 3}{9 \div 3} = \frac{1}{3}$. Again, .27,27, &c. $= \frac{27}{99} = \frac{27 \div 9}{99 \div 9} = \frac{3}{11}$.

RULE III. *When the given decimal is a mixed repeater or a mixed circulate, that is, with any finite part, from the mixed repeater or mixed circulate, subtract the finite part, and the remainder is the numerator of the vulgar fraction; the denominator is 9 for the repeating figure, or 9 for every figure of the circle, with as many cyphers annexed as there are figures in the finite part.*

Thus,

Thus, $.\dot{0}\dot{3} = \frac{3-0}{90} = \frac{3}{90} = \frac{3 \div 3}{90 \div 3} = \frac{1}{30}$; and $.0\dot{8}\dot{3}$ $= \frac{83-08}{900} = \frac{75}{900} = \frac{75 \div 75}{900 \div 75} = \frac{1}{12}$. Again, .0,45, $= \frac{45-0}{990} = \frac{45}{990} = \frac{45 \div 45}{990 \div 45} = \frac{1}{22}$; and .3,18, $= \frac{318-3}{990}$ $= \frac{315}{990} = \frac{315 \div 45}{990 \div 45} = \frac{7}{22}$; and .03,571428, $= \frac{3571428-03}{99999900} = \frac{3571425}{99999900} = \frac{3571425 \div 3571425}{99999900 \div 3571425}$ $= \frac{1}{28}$.

SECTION IX.

INVOLUTION.

DEFINITION. *Involution* is the finding of powers.

RULE. *Multiply the given, or first power, continually by itself, till the number of multiplications be* 1 *less than the index of the power to be found, and the last product will be the power required.*

Note. This Rule evidently comes from the definition of powers. See Introduction.

Examples.

I. What is the second power of 45?

Ans. $45 \times 45 = 2025$.

II

II. What is the ſquare of 4.16?

Anſ. $4.16 \times 4.16 = 17.3056$.

III. What is the third power of $\frac{2}{3}$?

Anſ. $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$.

IV. What is the third power of $1\frac{2}{3}$?

Anſ. $\frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} = \frac{125}{27}$; for $1\frac{2}{3} = \frac{5}{3}$.

SECTION X.

EVOLUTION.

DEFINITION. Evolution is the finding the roots of numbers either accurately or in decimals till the error be leſs than any propoſed number.

Note. A number is called a *complete power* of any kind when its root of the ſame kind can be accurately extracted; but if not, the number is called an *imperfect power*, and its root a *ſurd* or *irrational quantity*; ſo 4 is a complete power of the ſecond kind, its root being 2; but an imperfect power of the third kind, its third root being a ſurd quantity $= \sqrt[3]{4}$.

RULE. *Reſolve the given complete power into its ſingle diviſors or component parts; then take one from every two like diviſors, if the root is of the ſecond kind; from every three, if the root is of the third kind; from every four, if the root is of the fourth kind; and ſo on. The diviſors ſo taken and multiplied together will give the root required.*

Examples.

Examples.

I. $\sqrt{2025}=\sqrt{3\times3\times3\times3\times5\times5}=3\times3\times5=9\times5$ $=45.$

II. $\sqrt{17.3056}=\sqrt{2\times2\times2\times2\times2\times2\times2\times2\times2}$ $\times2\times13\times13=2\times2\times2\times2\times2\times13=4.16.$

III. $\sqrt[3]{\frac{8}{27}}=\sqrt[3]{\frac{2\times2\times2}{3\times3\times3}}=\frac{2}{3}.$

IV. $\sqrt[3]{\frac{125}{27}}=\sqrt[3]{\frac{5\times5\times5}{3\times3\times3}}=\frac{5}{3}=1\frac{2}{3}.$

V. $\sqrt[4]{265764994576}=\sqrt[4]{2.2.2.2.359.359.359}$ $.359=2.359=718.$

VI. $\sqrt[5]{248832}=\sqrt[5]{2.2.2.2.2.2.2.2.2.2.3.3.3.3.3}$ $=2.2.3=12.$

Note. We will give in another place the method of extracting the roots from the imperfect powers.

CHAP.

CHAPTER VI.

THE RULE OF EQUATIONS,

WHEREIN OF RATIOS, PROPORTIONS, PROGRESSIONS, AND INFINITE SERIES.

THE Rule of Equations is a method of finding theorems by the nature and properties of equations.

Principles. I. If you add equal quantities to both sides of an equation, or if you subtract equal quantities from both sides, the sums or differences will still continue equal to each other. As for instance, in this equation $12-4=8$, adding 4 to both sides, it will be $12-4+4=8+4$, that is, $12=12$. Again, subtracting 4 from both sides of the equation $12+4=16$, it will remain $12+4-4=16-4$, or $12=12$. Hence,

II. If a quantity be taken from either side of an equation, and placed on the other with a contrary sign, which is commonly called *transposition*, the two sides will be equal to each other. Thus, if $12+4=16$, transpose $+4$, and you will have $12=16-4$, or $12=12$; again, if $12-4=8$, transpose -4, and you will have $12=8+4$, or $12=12$.

III. If the two sides of an equation be multiplied or divided by the same number, the two products or quotients will still be equal to each other. Thus, if both sides of the equation $12\div3=4$ be multiplied into 3, we shall have $12\times3\div3=4\times3$, or $12=4\times3$, that is, $12=12$; and again, if both sides of the equation 4×3

$=12$ be divided by 3, it will be $4\times 3\div 3=12\div 3$, that is, $4=4$. Hence,

IV. Any quantity by which the firſt ſide of an equation is either multiplied or divided may be taken away by dividing or multiplying the other ſide by it. Thus, if $3\times 4=12$, we ſhall have $4=12\div 3$, or $4=4$; and again, if $12\div 3=4$, we ſhall have $12=4\times 3$, or $12=12$.

V. Every power or root of any given equation will ſtill give an equation. Thus, if $2\times 3=6$, it will be $2^2\times 3^2=6^2$ or $4\times 9=36$; and if $4\times 9=36$, it will be $\sqrt{4\times 9}=\sqrt{36}$, or $2\times 3=6$.

Rule of Equations. *Endeavour to expreſs the nature or relations of magnitude which is treated of by equations; then change theſe equations according to the principles now explained, and laſtly, reading the new equations you will find new verities or theorems.*

SECTION I.

OF RATIOS.

DEFINITIONS. I. The relation ariſing from the compariſon of quantities of the ſame kind is called *ratio*, and is of two kinds.

II. If we conſider the difference of the two quantities, it is called *arithmetical ratio*; but,

III. If we conſider their quotient, it is called *geometrical ratio*.

IV. The difference in the firſt caſe, or the quotient in the ſecond, is called the *exponent*.

V. The firſt quantity, or that which bears proportion, is called the *antecedent*; and,

VI.

VI. The other to which it bears proportion is called the *consequent*.

COROLLARIES. I. The arithmetical ratio betwixt 12 and 4 may be expressed by the equation $12-4=8$, because it shews the difference between 12 and 4.

II. The geometrical ratio betwixt 12 and 4 may be expressed by the equation $12\div4=3$, for it shews the quotient of 12 divided by 4.

Note. Geometrical ratio being most generally useful, is commonly called simply *ratio*.

I. *Of Arithmetical Ratio.*

PROBLEM. To find the common properties of an *arithmetical ratio.*

Solution. Let $12-4=8$ (Corollary I.) it will be $12=8+4$, and $12-8=4$ (Princ. II.) Therefore in arithmetical ratio

1. The difference between the antecedent and the consequent is equal to the exponent.

2. The sum of the exponent and consequent is equal to the antecedent.

3. The difference between the antecedent and exponent is equal to the consequent.

II. *Of Geometrical Ratio.*

DEFINITIONS. I. Two quantities are said to be in a *direct ratio*, when one increasing or decreasing, the other increases or decreases too. Thus, the work and the workmen are in a direct ratio.

II. Two quantities are said to be in an *inverse*, *reciprocal*, or *contrary ratio*, when one increasing or decreasing, the other on the contrary decreases or increases. Thus, the time and the workmen are in an inverse, reciprocal, or contrary ratio.

III. If the increase or decrease of a quantity depends either directly or inversely upon the increase or decrease of some others the first is said to be in a *compound ratio* either *direct* or *inverse* of the last quantities. Thus, the work is in a compound direct ratio of the time and workmen; but the time is in a compound ratio, direct of the work, and inverse of the workmen.

IV. Two ratios are called *direct* when their exponents are equal; so the two ratios $12 \div 4$ and $18 \div 6$ are direct, for their exponent is 3.

V. Two ratios are called *inverse*, *reciprocal*, or *contrary*, when their exponents are unequal, and on the contrary ratio to the other: so the $12 \div 4$ and $6 \div 18$ are inverse, reciprocal, or contrary, because their exponents $3 \div 1$ and $1 \div 3$ are unequal, and one the contrary ratio to the other.

VI. If a ratio results from the product of some others, it is called *compound*, and the others *simple* and *component*. Thus, the ratio $6 \div 1$ is compounded of the simple components $2 \div 1$ and $3 \div 1$, for $2 \div 1 \times 3 \div 1 = 6 \div 1$; and again, the same ratio is compounded of the simple components $10 \div 4$ and $12 \div 5$, for $10 \div 4 \times 12 \div 5 = 10 \times 12 \div 5 \times 4 = 2 \times 3 \div 1 \times 1 = 6 \div 1$.

VII.

VII. If two, three, four, or more component ratios are equal to one another, the compound of them is ſaid to be as *the ſquare, the cube, the biquadrate*, or other power of one of them. So the ratio $9 \div 1$ being compounded of the two ſimple equal ratios $3 \div 1$ and $3 \div 1$, is ſaid to be as the ſquare of 3; and the ratio $27 \div 1$ conſiſting of the three ſimple equal ratios $3 \div 1$, $3 \div 1$, and $3 \div 1$, is ſaid to be as the cube of 3.

Problem. To find the common properties of geometrical ratio.

Solution. Let $12 \div 4 = 3$ (Corollary II.) it will be $12 = 3 \times 4$, and $4 = 12 \div 3$ (Princ. IV.)

Therefore in geometrical ratio,

1. The antecedent divided by the conſequent quotes the exponent.

2. The product of the exponent into the conſequent gives the antecedent.

3. The antecedent divided by the exponent quotes the conſequent.

SECTION II.

OF PROPORTIONS.

DEFINITIONS. I. *Proportion* is the equality of two ratios, and is of two kinds.

II. If the two ratios are arithmetical, the proportion is *arithmetical:* but,

III. If they are geometrical, the proportion is *geometrical.*

IV. Four numbers are said to be *proportional*, when the ratio of the first to the second is the same as that of the third to the fourth.

V. Proportional numbers, or numbers in proportion, are usually denominated *terms*; of which the first and last are called *extremes*, and the intermediate ones *means*, or *middle terms*.

VI. If the mean terms are equal to one another, the proportion is called *continual*.

Corollaries. I. Let $12-4=8$, and $15-7=8$, the equation $12-4=15-7$ will express an arithmetical proportion. (Def. II.)

II. And, because four numbers arithmetically proportional are usually distinguished from one another, by writing them thus, $12 .. 4 :: 15 .. 7$, that is, 12 is to 4 as 15 to 7, this expression means the same thing as the foregoing equation $12-4=15-7$.

III. Again, let $12 \div 4=3$ and $18 \div 6=3$, the equation $12 \div 4=18 \div 6$ will represent a geometrical proportion. (Def. III.)

IV. And, because a geometrical proportion is usually expressed thus, $12 : 4 :: 18 : 6$, this expression and the preceding equation $12 \div 4=18 \div 6$ are one and the same thing.

V. The arithmetical proportion $12 .. 9 :: 9 .. 6$, and the geometrical $18 : 6 :: 6 : 2$, are continual (Def. VI.)

Note. Geometrical proportion being most generally useful, is commonly called simply *proportion* or *proportionality*, and its terms are also called only *proportionals*.

I. *Of Arithmetical Proportion.*

PROBLEM. To find the common properties of arithmetical proportion.

Solution. Let 12 .. 4 : : 15 .. 7 (Cor. II.) we ſhall find the following theorems,

I. $12-4=15-7$ (Cor. I. and II.) viz. of four quantities arithmetically proportional the difference of the firſt and ſecond is equal to the difference of the third and fourth.

II. $12+7=15+4$ (Princ. II.) viz. in four quantities arithmetically proportional, the ſum of the extremes is equal to the ſum of the means.

III. $7=15+4-12$ (Princ. II.) viz. when four quantities are arithmetically proportional, from the ſum of the means ſubtract the firſt, the remainder will be the fourth.

Again, let 12 .. 9 :: 9 .. 6 (Cor. V.) or $12-9=9-6$ (Cor. I.) it will be,

IV. $12+6=9+9=2\times 9$, viz. if three quantities be arithmetically proportional, the ſum of the extremes is double of the middle term.

V. $6=2\times 9-12$ (Princ. II.) viz. the third of three arithmetical proportionals is the difference betwixt double the ſecond and the firſt.

Corollaries. I. Of four arithmetical proportionals, any three being given, the fourth may be found by Theorem II.

II. Of three arithmetical proportionals, any two being given, the third may be found by Theorem IV.

II. *Of*

II. *Of Geometrical Proportion.*

PROBLEM. To find the common properties of geometrical proportion.

Solution. Let $12 : 4 :: 18 : 6$ (Cor. IV.) we shall find the following theorems,

I. $12 \div 4 = 18 \div 6$ (Cor. III.) viz. of four quantities geometrically proportional, the quotient of the first and second is equal to the quotient of the third and fourth.

II. $12 \times 6 = 18 \times 4$ (Princ. IV.) viz. the product of the extremes of four quantities geometrically proportional is equal to the product of the means; and conversely.

III. $6 = 18 \times 4 \div 12$ (Princ. IV.) viz. If the product of the mean proportionals be divided by the first term, the quotient will give the fourth.

IV. Being $12 \times 6 = 18 \times 4$, it is also $2 \times 12 \times 6 \times 3 = 2 \times 18 \times 4 \times 3$ (Princ. III.) and then $2 \times 12 : 4 \times 3 :: 2 \times 18 : 6 \times 3$ (Cor. IV.) or $24 : 12 :: 36 : 18$, viz. any equimultiples whatever of the first and third proportionals have the same ratio, as any equimultiples whatever of the second and fourth.

V. For the same reason $12 \times 6 \div 3 \times 2 = 18 \times 4 \div 3 \times 2$ (Pr. III.) and therefore $12 \div 3 : 4 \div 2 :: 18 \div 3 : 6 \div 2$ (Cor. IV.) or $4 : 2 :: 6 : 3$, viz. any parts whatever of the first and third proportionals have the same ratio, as any parts whatever of the second and fourth.

VI. Because $12 \times 6 = 18 \times 4$, it is likewise $12^2 \times 6^2 = 18^2 \times 4^2$, $12^3 \times 6^3 = 18^3 \times 4^3$, &c. and $\sqrt{12 \times 6} = \sqrt{18 \times 4}$, $\sqrt[3]{12 \times 6} = \sqrt[3]{18 \times 4}$, &c. (Princ. V.) and then $12^2 : 4^2 :: 18^2 : 6^2$, $12^3 : 4^3 :: 18^3 : 6^3$, &c. and $\sqrt{12} : \sqrt{4} :: \sqrt{18} : \sqrt{6}$, $\sqrt[3]{12} : \sqrt[3]{4} :: \sqrt[3]{18} : \sqrt[3]{6}$,

&c.

&c. (Cor. IV.) viz. if four quantities be proportional, their alike powers and roots are also proportional.

VII. $12 \times 6 = 18 \times 4$, then $\frac{12}{18} = \frac{4}{6}$ (Princ. IV.) or $12 : 18 :: 4 : 6$ (Cor. IV.) which is called *alternation.*

VIII. $\frac{4}{12} = \frac{6}{18}$ (Princ. IV.) that is, $4 : 12 :: 6 : 18$ (Cor. IV.) *inverse proportion.*

IX. $12 \times 6 + 4 \times 6 = 18 \times 4 + 4 \times 6$ (Principle I.) or $\frac{12+4}{4} = \frac{18+6}{6}$ (Pr. IV.) i. e. $12+4 : 4 :: 18+6 : 6$ (Cor. IV.) proportion by *composition.*

X. $12 \times 6 - 4 \times 6 = 18 \times 4 - 4 \times 6$ (Principle I.) or $\frac{12-4}{4} = \frac{18-6}{6}$ (Pr. IV.) i. e. $12-4 : 4 :: 18-6 : 6$ (Cor. IV.) proportion by *division.*

XI. $12 \times 18 - 12 \times 6 = 12 \times 18 - 4 \times 18$ (Princ. I.) or $12 \times \overline{18-6} = 18 \times \overline{12-4}$, then $\frac{12}{12-4} = \frac{18}{18-6}$ (Pr. IV.) that is, $12 : 12-4 :: 18 : 18-6$ (Cor. IV.) proportion by *conversion.*

Again, let $18 : 6 :: 6 : 2$, we shall have

XII. $18 \times 2 = 6 \times 6$ (by the foregoing Theorem II.) $= 6^2$, viz. in continual proportion, the product of the extremes is equal to the square of the middle term.

XIII. $2 = 6^2 \div 18$, viz. the third of three proportionals is equal to the square of the second divided by the first.

XIV. $12 \div 4 = 18 \div 6$, $18 \div 6 = 6 \div 2$, then $12 \div 4 = 6 \div 2$, or $12 : 4 :: 6 : 2$, viz. two ratios equal to a third must be equal to one another and make a proportion.

Lastly, let $12 : 4 :: 18 : 6$, and $5 : 8 :: 10 : 16$, then,

XV. $\frac{12}{4} = \frac{18}{6}$ and $\frac{5}{8} = \frac{10}{16}$, therefore $\frac{12 \times 5}{4 \times 8} = \frac{18 \times 10}{6 \times 16}$ (Principle III.) or $12 \times 5 : 4 \times 8 :: 18 \times 10 : 6 \times 16$ (Cor. IV.) viz. two proportions multiplied one by another give a new proportion.

P XVI.

XVI. By the same reason $\frac{12}{4} \div \frac{5}{8} = \frac{18}{6} \div \frac{10}{16}$ (Princ. III.) or $\frac{12}{4} \times \frac{8}{5} = \frac{18}{6} \times \frac{16}{10}$, or $\frac{12}{5} \times \frac{8}{4} = \frac{18}{10} \times \frac{16}{6}$, or $\frac{12}{5} \times \frac{6}{10} = \frac{18}{10} \times \frac{4}{8}$ (Pr. IV.) i. e. $\frac{12}{5} : \frac{4}{8} :: \frac{18}{10} : \frac{6}{16}$ (Cor. IV.) viz. two proportions divided one by another give a new proportion.

Corollaries. I. A proportion gives always an equation, whereof either side will be the product of the extremes and the other side the product of the middle terms (Theor. II.).

II. Whenever we have an equation wherein the product of two quantities on one side is found equal to the product of two quantities on the other, such an equation may be resolved into four proportionals, by making the two quantities on either side the extremes, and those on the other side the middle terms.

SECTION III.

OF PROGRESSIONS.

DEFINITIONS. I. Any number of quantities continually proportional is called *Progression.*

II. If the quantities increase or decrease by the same common difference, the progression is *arithmetical*, as 1, 2, 3, 4, 5, &c. and 8, 6, 4, 2, &c.

III. If the quantities increase by the same multiplier or decrease by the same divisor, the progression is *geometrical*, as 1, 2, 4, 8, 16, &c. and 16, 8, 4, 2, &c.

IV.

IV. This common multiplier or divisor is called the *common ratio.*

I. *Of Arithmetical Progression.*

PROBLEM. To find the fundamental property of arithmetical progression.

Solution. I take the natural progression 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c.

Because 1..2::11..12, it will be $1+12=2+11$, that is,

1..3::10..12	$1+12=3+10$
1..4:: 9..12	$1+12=4+9$
1..5:: 8..12	$1+12=5+8$
1..6:: 7..12	$1+12=6+7$

In an arithmetical progression, the sum of the first and last terms is equal to the sum of any two intermediate terms, equally distant from the extremes.

Corollary. Hence
$$\left.\begin{array}{l} 1+12=13 \\ 2+11=13 \\ 3+10=13 \\ 4+9=13 \\ 5+8=13 \\ 6+7=13 \end{array}\right\} = 13\times 6$$
that is:

The sum of all the terms of an arithmetical progression, is equal to the sum of the extremes taken half as often as there are terms.

Example. Required the sum of 50 terms of the series 2, 4, 6, 8, &c. The first term being 2 and the last 100, the sum required will be $=\overline{2+100}\times 50\div 2 = 102\times 25 = 2550$.

II. *Of Geometrical Progreſſion.*

PROBLEM. I. To find the fundamental property of geometrical progreſſion.

Solution. Take the progreſſion 1, 2, 4, 8, 16, 32, 64, 128.

Being $1:2::64:128$, it will be $1\times128=2\times64$, that is
$1:4::32:128$ $\quad 1\times128=4\times32$
$1:8::16:128$ $\quad 1\times128=8\times16$

The product of the extremes in a geometrical progreſſion, is equal to the product of any two terms equally diſtant from the extremes.

Problem II. The extremes and common ratio being given, to find the ſum of the progreſſion.

Solution. By the compound diviſion, $\frac{1}{1-2}=1+2+4+8+16+\frac{32}{1-2}$; and therefore $\frac{1}{1-2}-\frac{32}{1-2}=1+2+4+8+16$. Now $\frac{-32}{1-2}=\frac{-32}{-2+1}=16-\frac{16}{-2+1}=16-\frac{16}{1-2}$; then $1+2+4+8+16=\frac{1}{1-2}-\frac{32}{1-2}=\frac{1}{1-2}-\frac{16}{1-2}+16=\frac{1-16}{1-2}+16=\frac{-1+16}{-1+2}+16=\frac{15}{1}+16=31$. Hence divide the difference of the extremes by the difference of the common ratio and unity, and the quotient added to the greateſt term gives the ſum required.

Ex-

Example. A gentleman who had a daughter married on new year's day, gave the husband towards her portion 4 shillings, promising to triple that sum the first day of every month, for nine months after the marriage; the sum paid on the first day of the ninth month was 26244 shillings: what was the lady's portion?

Ans. $\frac{26244-4}{3-1}+26244=\frac{26240}{2}+26244=13120$ $+26244=39364s.$ or $1968l.\ 4s.$

SECTION IV.

OF SERIES.

DEFINITIONS. I. *Series* is a rank of numbers increasing, or decreasing, without interruption, according to a certain law.

II. A fraction containing this law, may be called the *scale* of a series.

Probl. I. The scale of a series being given, to find this series.

Solution. Perform the division, and the quotient will be the series.

Example. What is the series, the scale of which is $\frac{1}{1-4+4}$?

$$
\begin{array}{l}
1-4+4)\,1\quad(1+4+12+32+80+\dfrac{192-320}{1-4+4} \\
\quad 1-4+4 \\
\hline
\qquad 4-4 \\
\qquad 4-16+16 \\
\hline
\qquad\quad 12-16 \\
\qquad\quad 12-48+48 \\
\hline
\qquad\qquad 32-48 \\
\qquad\qquad 32-128+128 \\
\hline
\qquad\qquad\quad 80-128 \\
\qquad\qquad\quad 80-320+320 \\
\hline
\qquad\qquad\qquad 192-320=-128 \\
\qquad\qquad\qquad \text{\&c.}\quad \text{\&c.}
\end{array}
$$

Probl. II: The scale of a series being given, to find the sum of any number of its terms.

Solution. Because $\frac{1}{1-4+4}=1+4+12+32+80$ $+\frac{192-320}{1-4+4}$, it follows that $1+4+12+32+80=$ $\frac{1}{1-4+4}-\frac{192+320}{1-4+4}=\frac{321-192}{1-4+4}=\frac{129}{1}=129.$

Theorem. Subtract the remainder of the division from the dividend; then divide this new remainder by the divisor, and the quotient will be the sum required. Thus, in the same example, we find $\frac{1+128}{1}=129.$

Another Example. What is the sum of seven terms of the series 2, 4, 2, −8, −22, −20, 26, its scale being $\frac{2}{1-2+3}$?

Ans.

Anſ. Becauſe the remainder after ſeven terms or diviſions is $112-78=34$ the ſum required is $=\frac{2-34}{1-2+3}$ $=\frac{-32}{2}=-16$.

Note. I do not purſue the different problems belonging to ſeries conſidered in this point of view, becauſe it would be ſufficient matter for an arithmetical Treatiſe, and therefore does not ſuit my preſent purpoſe. In relation to the arithmetical and geometrical ſeries or progreſſions, I will take their properties again into conſideration when I treat of the general theorems in the algebraical Language.

CHAPTER VII.

THE RULE OF THREE.

THE *Rule of Three*, called alſo on account of its excellence the *Golden Rule*, from certain numbers given, finds another; and is divided into *ſimple* and *compound*, or into *ſingle* and *double*.

The *ſimple* or *ſingle* Rule of Three, from three numbers given, finds a fourth, to which the third bears the ſame ratio as the firſt does to the ſecond.

The *compound* or *double* Rule of Three, from five given numbers, finds a ſixth, or from ſeven given numbers finds an eighth, or from nine given numbers, finds a tenth, &c.

The Rule of Three is alſo diſtinguiſhed into *direct* and *inverſe* or *reciprocal*.

The Rule of Three is ſaid to be *direct* when the firſt bears the ſame ratio to the ſecond as the third does to the fourth, in which caſe the greater the ſecond term is

in

in respect to the first, the greater will the fourth term be in respect to the third, and the contrary. Thus, 4:10::16:40 and 40:16::10:4 are two direct proportions, because in the first 10 being greater than 4, 40 is proportionably greater than 16, and in the second 16 being less than 40, 4 is of consequence less than 10 in the same ratio.

Hence in direct proportion the product of the extremes will always be equal to that of the means. Thus, $4 \times 40 = 10 \times 16$, and $40 \times 4 = 16 \times 10$.

The Rule of Three is said to be *inverse* or *reciprocal*, when the third bears the same ratio to the first as the second does to the fourth, in which case the less the third term is in respect to the first, the greater will the fourth term be in respect to the second, and *vice versa*. For instance, suppose 8 men could do a certain piece of work in 4 days, and it were required to know in what time 16 men could do it; upon the least consideration it would occur, that 16 hands would do more work than 8, and consequently require less time to do the same work, wherefore, as 8:4::16:2, does signify that 16 bear the same ratio to 8, that 4 does to 2.

Hence in inverse proportion the product of the two first terms will always be equal to the product of the two last, for $8 \times 4 = 16 \times 2$.

Every question in the Rule of Three may be divided into two parts, viz. a *supposition* and a *demand*; and of the three given numbers two are always found in the supposition and only one in the demand.

RULES. I. *Place that term of the supposition, which is of the same kind with the number sought, in the middle. The two remaining terms are extremes and always of the same kind; but make that number the third term upon which the demand lies.*

II. *Consider, from the nature of the question, whether the proportion must be direct or inverse.*

III.

III. *If the terms must be in direct proportion, that is, if more require more, or less require less, the product of the two last divided by the first will quote the answer or fourth proportional.*

IV. *If the terms must be in inverse proportion, that is, if more require less, or less require more, the product of the two first divided by the last will quote the answer.*

Note. We treat here only of the simple Rule of Three, reserving the compound, for facility's sake, to Section III. of this Chapter.

SECTION I.

THE SIMPLE RULE OF THREE DIRECT.

Quest. I. If 4 yards cost 12 shillings, what will 6 yards cost at that rate?

In this question the supposition is, if 4 yards cost 12 shillings; and the two terms contained in it are 4 yards and 12 shillings; the demand lies in these words, what will 6 yards cost? and the only term found in it is 6 yards. Next the number sought is the price of 6 yards, and the term in the supposition of the same kind is the price of 4 yards, viz. 12 shillings, which I place in the middle, as directed in Rule I. and the two remain-

```
    yds.  s.   yds.
If  4  :  12 : : 6 : ?
           6
          ——
 4 ) 72 ( 18
     4
    ——
     32
     32
    ——
      0
```

ing terms are extremes, and of the same kind, viz. both length; but term 6 yards being that upon which the demand lies, it stands, by the same rule, in the third place, as in last page. Lastly, the third term, 6 yards, contains a greater quantity than the first term, 4 yards, and consequently requires a greater price, therefore, I conclude the terms to be in a direct proportion, and find the answer by dividing the product of the two last terms by the first, Rule III. as in the preceding page.

Quest. II. If 7 *cwt*. of pepper cost 21*l*. how much will 5 *cwt*. cost at that rate?

cwt. *l*. *cwt*.
$7 : 21 :: 5 : 5 \times 21 \div 7 = 5 \times 3 = 15$ pounds.

Quest. III. If 13 *yds*. of velvet cost 21*l*. what will 27 *yds*. cost at that rate?

yds. *l*. *yds*.
$13 : 21 :: 27 : 21 \times 27 \div 13 = 567 \div 13 = 43\frac{8}{13} =$
43*l*. 12*s*. 3*d*. 2*f*. $\frac{10}{13}$.

yds. *l*. *yds*.
13 : 21 : : 27
27
—— ×
147
42
—— +
13) 567 (43
52
——
47
39
——
rem. 8 *l*.

l.
8
20
—— ×
13) 160 (12*s*.
13
——
30
26
——
rem. 4 *s*.

4*s*.

s.	d.
4	9
12 ×	4 ×
13) 48 (3d.	13) 36 (2f.
39 ×	26 −
rem. 9d.	rem. 10f.

Quest. IV. If 14*lb.* of tobacco cost 27*s.* what will 478*lb.* cost at that rate?

lb.   *s.*   *lb.*

$14 : 27 :: 478 : 478 \times 27 \div 14 = 12906 \div 14 = 6453 \div 7 = 921\frac{6}{7}s. = 46l.\ 1s.\ 10d.\ 1f.\ \frac{1}{7}.$

Quest. V. If 18 *cwt.* of sugar cost 54*l.* what wil 7 *cwt.* 3 *qrs.* 14 *lb.* cost at that rate?

cwt.   *l.*   *cwt. q. lb.*

18 : 54 :: 7 3 14 :

18	7	882
112 ×	112 ×	54 ×
36	14	3528
18 .	77 .	4410 +
18 . . +	784*lb.* +	2016) 47628 (23*l.*
2016*lb.*	84*lb.* = 3*qrs.*	4032 −
	14 +	7308
	882	6048 −
		rem. 1260

```
            1260
              20
          ------
2016 )  25200 ( 12s.
        2016
        ----
         5040
         4032
        -----
rem.    1008 s.
          12
        -----
2016 )  12096 ( 6d.
        12096
        -----
           0
```

Anf. 23*l.* 12*s.* 6*d.*

Queſt. VI. If 3 *cwt.* 1 *qr.* 14 *lb.* of raiſins coſt 10*l.* 2*s.* 6*d.* what will 6 *cwt.* 3 *qrs.* coſt at that rate?

cwt.	*qr.*	*lb.*		*l.*	*s.*	*d.*		*cwt.*	*qrs.*		
3	1	14	:	10	2	6	::	6	3	: ?	or,

lb.		*d.*		*lb.*		*d.*	*lb.*	*lb.*	*l.*	*s.*
378	:	2430	::	756	:	2430 ×	756 ÷	378 =	20	5

Queſt. VII. If A can finiſh a work in 20 days, and B in 30 days; in what time will the work be finiſhed by A and B working together?

da. *work.* *da.* *work.*

$20 : 1 :: 30 : \frac{30}{20} = \frac{3}{2} = 1.5$ done by A, in 30 days.

1 done by B, in 30 days.

2.5 done by A and B, in 30 days.

work. *da.* *work.* *da.*

$2.5 : 30 :: 1 : \frac{30}{2.5} = \frac{30.0}{2.5} = 12$

Queſt,

Quest. VIII. A ciſtern in a certain conduit is ſupplied with water by one pipe of ſuch bigneſs, that if the cock A at the end of the pipe be ſet open, the ciſtern will be filled in ½ an hour; but at the bottom of the ciſtern two other cocks B and C are placed, whoſe capacities are ſuch, that by the cock B ſet open alone, all the reſt being ſtopped, the ciſtern ſuppoſed to be full, will be emptied in $1\frac{3}{7}$ or $\frac{10}{7}$ hour; alſo, by the cock C ſet open alone, the ciſtern will be emptied in $2\frac{1}{3}$ or $\frac{7}{3}$ hour. Now, becauſe more water will be infuſed by the cock A, than can be expelled by both the cocks B and C in one and the ſame time: the queſtion is to find in what time the ciſtern will be filled, if all the ſaid three cocks be ſet open at once?

ho. *ciſt.* *ho.* *ciſt.*

$$\frac{10}{7} : 1 :: 1 : \frac{7}{10} = \frac{7\times 7}{10\times 7} = 49 \div 70 \quad \text{emptied by C in an hour.}$$

$$\frac{7}{3} : 1 :: 1 : \frac{3}{7} = \frac{3\times 10}{7\times 10} = 30 \div 70 \quad \text{emptied by B in an hour.}$$

$$\underline{}\,+ \qquad 79 \div 70 \quad \text{emptied by B \& C in an hour.}$$

$$\frac{1}{2} : 1 :: 1 : 2 = \frac{2\times 70}{70} = 140 \div 70 \text{ filled by A in 1 h.}$$

$$*\,-\underline{}$$

$$61 \div 70 \text{ gained by A in 1 h.}$$

c. *h.* *c.* *h.*

$$\frac{61}{70} : 1 :: 1 : \frac{70}{61} = 1\tfrac{9}{61}. \text{ Anſ.}$$

* When the mark of Subtraction (—) is before the ſtraight line, the ſuperior number ought to be ſubtracted from the inferior.

Queſt.

Quest. IX. Suppose a dog, a wolf, and a lion, were to devour a sheep, and that the dog could eat up the sheep in an hour, the wolf in $\frac{3}{4}$ of an hour, and the lion in $\frac{1}{2}$ an hour. Now, if the lion begin to eat $\frac{1}{8}$ of an hour before the other two, and afterwards all three eat together, the question is, in what time the sheep would be devoured?

ho. sh. ho. sh.

$\frac{1}{2} : 1 :: \frac{1}{8} : \frac{1}{8} \div \frac{1}{2} = \frac{2}{8} = \frac{1}{4}$ devoured by the lion in $\frac{1}{8}$ of an hour, and therefore $\frac{3}{4}$ of the sheep only remains.

$$\left.\begin{array}{l} \frac{1}{2} : 1 :: 1 : 2 = 6 \div 3 \\ \frac{3}{4} : 1 :: 1 : \frac{4}{3} = 4 \div 3 \\ \qquad\qquad\quad 1 = 3 \div 3 \end{array}\right\} \begin{array}{c} \text{devour-} \\ \text{ed by} \\ \text{the} \end{array} \left\{\begin{array}{l} \text{lion} \\ \text{wolf} \\ \text{dog} \end{array}\right\} \text{in an hour.}$$

$13 \div 3$ devoured by all three together in an hour.

sh. h. sh. h.

$\frac{13}{3} : 1 :: \frac{3}{4} : \frac{3}{4} \div \frac{13}{3} = \frac{9}{52} = 18 \div 104$, time of the common eating.

$\frac{1}{8} = \frac{1 \times 13}{8 \times 13} = 13 \div 104$, time the lion was eating alone.

$31 \div 104$, time required, which is not quite 18 minutes.

Quest. X. A certain footman A sets out from London towards Lincoln, and at the same time another footman B departs from Lincoln towards London; also A travels every day $2\frac{1}{2}$ miles more than B. Now, supposing those two cities to be 100 miles distant one from the other, and that those two footmen do meet one another at the end of 8 days after the beginning of their journies; the question is, how many miles each will have then travelled, as also how many miles each travelled daily?

da. 1

da. mi. da. mi.
1 : $2\frac{1}{2}$ :: 8 : 20, which A had travelled more than B, in 8 days.
100, distance between London and Lincoln.

80
—÷2
40, which B had travelled.
20
—+
60, which A had travelled.

da. mi. da. mi.
8 : 60 :: 1 : $60 \div 8 = 7\frac{1}{2}$ travelled daily by A.
8 : 40 :: 1 : $40 \div 8 = 5$ travelled daily by B.

Quest. XI. There is an island which is 134 miles in compass; now at the same time, and from the same place, two travellers A and B begin a journey round the said island, but they travel contrary ways at this rate, viz. A travels 11 miles every 2 days, and B 17 miles every 3 days; the question is, to find in what space of time A and B will meet one another, and how many miles each will then have travelled?

da. mi. da. mi.

$2 : 11 :: 1 : \frac{11}{2} = \frac{11 \times 3}{2 \times 3} = 33 \div 6$, A trav. in 1 day.

$3 : 17 :: 1 : \frac{17}{3} = \frac{17 \times 2}{3 \times 2} = 34 \div 6$, B trav. in 1 day.

—— +
$67 \div 6$, trav. A & B in 1d.

$\frac{67}{6} : 1 :: 134 : \frac{134 \times 6}{67} = \frac{2 \times 6}{1} = 12$, time when A and B will meet one another.

$2 : 11 :: 12 : \frac{11 \times 12}{2} = \frac{11 \times 6}{1} = 66$, travelled by A.

$3 : 17 :: 12 : \frac{17 \times 12}{3} = \frac{17 \times 4}{1} = 68$, travelled by B.

Quest.

Quest. XII. There is another island which is 36 miles in compass; now, if at the same time, and from the same place, two travellers A and B set forward to travel round the said island, and follow one another in such manner, that A travels every day 9 miles and B 7 miles; the question is, to find in what time they will meet again, also how many miles, and how many times about the island each person will then have travelled?

$9 - 7 = 2$ miles gained by A.

mi. *da.* *mi.* *da.*
2 : 1 :: 36 : 18, time necessary for A to gain over B, the whole compass of the island.

18 *days.*	18 *days.*
9 *miles.*	7 *miles.*
36) 162 (4½	36) 126 (3½

Therefore they will meet at the end of 18 days from their first parting; and then A will have travelled 162 miles, or 4½ times the compass of the island; and B will have travelled 126 miles, or 3½ the compass of the island.

SEC-

SECTION II.

THE SIMPLE RULE OF THREE INVERSE.

QUESTION I. If 8 men can do a piece of work in 12 days, in how many days will 16 men do the ſame?

In this queſtion it is obvious that more requires leſs, for 16 men will do the work in fewer days than 8 men, and therefore the proportion being inverſe, the product of the two firſt terms divided by the laſt or third, will quote the anſwer as follows:

$$\begin{array}{ccccccc} m. & & d. & & m. & & d. \\ 8 & : & 12 & :: & 16 & : & \frac{12\times 8}{16}=\frac{12\times 8}{2\times 8}=6 \text{ days.} \end{array}$$

Queſt. II. If 30 yards of cloth that is 5 quarters broad be required to hang a bed, how many yards of 3 quarters broad will ſerve the ſame purpoſe?

$$\begin{array}{ccccccc} \text{Breadth.} & & \text{Length.} & & \text{Breadth.} & & \text{Length.} \\ q. & & y. & & q. & & y. \\ 5 & : & 30 & :: & 3 & : & \frac{30\times 5}{3}=\frac{10\times 5}{1}=50 \text{ yds.} \end{array}$$

Queſt. III. If 360 men be in garriſon, and have proviſions only for 6 months, how many men muſt be turned out that the ſame ſtock of proviſions may laſt 9 months?

Anſ. 240 men are to be retained, and the reſt, viz. 120 muſt be turned out.

SECTION III.

THE COMPOUND OR DOUBLE RULE OF THREE, EITHER DIRECT OR INVERSE.

IN any queſtion of this Rule there are always two or more numbers in the ſuppoſition of the ſame name, or ſignifying the ſame thing, with two or more numbers in the demand; and in both the ſuppoſition and the demand there are ſome terms which may be conſidered as *cauſes*, and ſome others which may be conſidered as *effects*.

RULES. I. *Place the terms of the ſuppoſition and thoſe of the demand in two parallel lines, writing the terms under each other which are of the ſame name, and putting inſtead of the unknown term the point of interrogation, thus (?).*

II. *Multiply together, in each of theſe lines, all the terms which ſtand for the cauſes, in order to have* two cauſes; *and alſo all the terms which ſtand for the effects, to have* two effects. *Then,*

III. *Becauſe the effects are proportional to their cauſes, the ſaid two cauſes and two effects will furniſh a proportion, in which the firſt cauſe will be to the firſt effect, as the ſecond cauſe is to the ſecond effect.*

IV. *Take the product of the extremes and that of the means. Laſtly,*

V. *Divide the product, in which no term is wanting, by that in which the unknown term is wanted, or the point of interrogation is contained, and the quotient will be the anſwer.*

Queſt.

Queſt. I. To find how many men can compleat a trench of 135 yards long in 8 days, it being known that 16 men can dig 54 yards in 6 days?

men.	*ya.*	*da.*		Cauſes.		Effects.
16	54	6	1ſt	16×6	1ſt	54
?	135	8	2d	$? \times 8$	2d	135

Proportion $16 \times 6 : 54 :: ? \times 8 : 135$.

Anſwer $\frac{16 \times 6 \times 135}{54 \times 8} = 30$ men.

Queſt. II. If a footman travel 130 miles in 3 days when the days are 12 hours long, in how many days of 10 hours each may he travel 650 miles?

mi.	*da.*	*ho.*		Cauſes.		Effects.
130	3	12	1ſt	3×12	1ſt	130
650	?	10	2d	$? \times 10$	2d	650

Proportion $3 \times 12 ; 130 :: ? \times 10 : 650$.

Anſwer $\frac{3 \times 12 \times 650}{130 \times 10} = 18$ days.

Queſt. III. If 600 ſeamen in 1 week eat 13 *cwt.* 1 *q.* 16 *lb.* of beef, how many lb. will ſerve 120 ſeamen 12 weeks?

ſeam.	*w.*	*lb.*		Cauſes.		Effects.
600	1	1500	1ſt	600×1	1ſt	1500
120	12	?	2d	120×12	2d	?

Proportion $600 \times 1 ; 1500 :: 120 \times 12 : ?$

Anſwer $\frac{1500 \times 120 \times 12}{600 \times 1} = 3600$ lb.

Queſt. IV. If the price of 10 *oz.* of bread, when corn is at 4*s.* 3*d.* per buſhel, be 3½*d.* what muſt be paid for 2*lb.* 3*oz.* when the corn is 5*s.* per buſhel?

oz.	*p. co.*	*p. br.*		Cauſes.		Effects.
10	51*d.*	3½*d.*	1ſt	10 × 51	1ſt	3½
27	60*d.*	?	2d	27 × 60	2d	?

Proportion $10 \times 51 : 3\frac{1}{2} :: 27 \times 60 : ?$

Anſwer $\frac{3\frac{1}{2} \times 27 \times 60}{10 \times 51} = 11\frac{2}{17}d.$

Queſt. V. If 7 *oz.* 5 *dwts.* of bread be bought at 4¾*d.* when corn is at 4*s.* 2*d.* per buſhel, what weight of it may be bought for 1*s.* 2*d.* when the price of the buſhel is 5*s.* 6*d.*

dwts.	*p. br.*	*p. co.*		Cauſes.		Effects.
145	4¾*d.*	50*d.*	1ſt	145 × 50	1ſt	4¾
?	14*d.*	66*d.*	2d	? × 60	2d	14

Prop. $145 \times 50 : 4\frac{3}{4} :: ? \times 66 : 14.$

Anſ. $\frac{145 \times 50 \times 14}{4\frac{3}{4} \times 66} = 323$*dwts.* or 1*lb.* 4*oz.* $3\frac{479}{627}$*dwts.*

Queſt. VI. A wall, which was to be built to the height of 27 feet, was raiſed to the height of 9 feet by 12 men in 6 days; how many men muſt be employed to finiſh the wall in 4 days, at the ſame rate of working?

bei.

hei.	*me.*	*da.*		Causes.		Effects.	
9*f.*	12	6	1st	12×6		1st	9
18*f.*	?	4	2d	$? \times 4$		2d	18

Prop. $12 \times 6 : 9 :: ? \times 4 : 18.$

Ans. $\frac{12 \times 6 \times 18}{9 \times 4} = 36$ men.

Quest. VII. If 500 men working 14 hours a day can do a wall of 540 feet in length in 25 days, in how many days may 1400 men perform 1296 feet of the same wall working only 12 hours a day?

me.	*ho.*	*da.*	*ft.*
500	14	25	540
1400	12	?	1296

	Causes.	Effects.	
1st	$500 \times 14 \times 25$	1st	540
2d	$1400 \times 12 \times ?$	2d	1296

Prop. $500 \times 14 \times 25 : 540 :: 1400 \times 12 \times ? : 1296.$

Ans. $\frac{500 \times 14 \times 25 \times 1296}{540 \times 1400 \times 12} = 25$ days.

CHAP.

CHAPTER VIII.

THE RULE OF EXAMPLES.

The Rule of Examples takes an example or a particular case of the question proposed, in order to find a general answer to it.

Rule of Examples. *Take any numbers at pleasure, and perform the same operations with them, as in the question are described to be performed with the numbers sought, endeavouring to express their given relations by way of equations or proportions, then work these equations according to their principles, in order to find a general rule for the answer required.*

Quest. I. Two factors of the number 768 are to one another as 3 is to 4; I demand these factors.

I take two numbers, for instance, 6 and 8, which are to one another as 3 is to 4, and multiplied together produce 48. Because $6=3\times2$, and $8=4\times2$, it is $6\times8=3\times2\times4\times2=3\times4\times2^2=12\times4$, and consequently $48=12\times4$; whence $48\div12=4$. Now, it is obvious, that 12 is the product of the given numbers 3 and 4, which express the ratio of the factors 6 and 8 of the number 48; and it appears likewise that the quotient 4, arising from the division of 48 by 12, is the square of the number 2, that shews the ratio in which the factors 6 and 8 are multiples of the given numbers 3 and 4. Hence arises the following

RULE.

RULE. *Divide the given number by the product of the terms which express the ratio of its factors, and extract the square root from the quotient; then multiply severally each term of that ratio by this root, and the two products will give the two factors required.*

Examples.

I. In our question $768 \div 12 = 64$, and $\sqrt{64} = 8$; then $3 \times 8 = 24$ and $4 \times 8 = 32$. The factors required are therefore 24 and 32, as must be, for $24 : 32 :: 3 : 4$ and $24 \times 32 = 768$.

II. What are the factors of 80, if their ratio is as 12 to 15?

$\frac{80}{12 \times 15} = \frac{80}{180} = \frac{8}{18} = \frac{4}{9}$, and $\sqrt{\frac{4}{9}} = \frac{2}{3}$; then $12 \times \frac{2}{3} = 8$ and $15 \times \frac{2}{3} = 10$. Therefore the factors required are 8 and 10; really $8 \times 10 = 80$, and $8 : 10 :: 12 : 15$.

III. If the factors of the same number 80 be as 5 is to 8, it will be $\frac{80}{5 \times 8} = \frac{80}{40} = 2$, whose square root $= \sqrt{2}$; consequently the factors required are $5 \times \sqrt{2} = 5\sqrt{2}$, and $8 \times \sqrt{2} = 8\sqrt{2}$, for $5\sqrt{2} \times 8\sqrt{2} = 40\sqrt{2} \times \sqrt{2} = 40\sqrt{4} = 40 \times 2 = 80$, and $5\sqrt{2} : 8\sqrt{2} :: 5 : 8$, because $8 \times 5\sqrt{2} = 5 \times 8\sqrt{2}$.

Quest. II. To find such a number which increased by 20 be to itself as 9 to 7.

Add, for instance, 6 to 20, and you will find $26 : 6 :: 13 : 3$; then by division $26 - 6 : 6 :: 13 - 3 : 3$, that is, $20 : 6 :: 10 : 3$, or $10 : 3 :: 20 : 6$. The number 6 is

6 is therefore the fourth proportional after 10, 3, and 20. Now, 10 shews the difference between the terms of ratio $13 \div 3$ taken at pleasure, 3 is the consequent of the same ratio, and 20 is the given number. Then we have the following general

RULE. *Take the difference betwixt the terms of the proposed ratio, then make a proportion, whereof the first term is this difference, the second the consequent of the same ratio, the third the given number, and the fourth will be the number sought.* Thus, in our question

$$9-7=2, \text{ and } 2 : 7 :: 20 : \frac{20\times 7}{2}=\frac{10\times 7}{1}=70.$$

For $20+70 : 70 :: 9 : 7$, or $90 : 70 :: 9 : 7$, as it is required.

Quest. III. The difference between 20 and a certain number is to this number as 3 is to 2 : what is that number?

Suppose the number sought to be 6, and being therefore $20-6=14$ you will have $14 : 6 :: 7 : 3$; then by composition, $14+6 : 6 :: 7+3 : 3$, that is, $20 : 6 :: 10 : 3$, or $10 : 3 :: 20 : 6$. But 10 being the sum of the terms of ratio $7 \div 3$ taken at pleasure, 3 the consequent of this ratio, and 20 the given number, we find the following

RULE. *The number sought is the fourth proportional after the sum of the terms of the proposed ratio, the consequent of the same ratio, and the given number.* Thus the number sought in the question is 8, because

$$3+2=5, \ 5 : 2 :: 20 : \frac{20\times 2}{5}=\frac{4\times 2}{1}=8, \text{ and}$$

$20-8 : 8 :: 3 : 2$, or $12 : 8 :: 3 : 2$, as required by the question.

Quest.

Quest. IV. Achilles follows a tortoise at the interval of a mile; their velocities are as 100 to 1: at what distance will he overtake the tortoise?

It is obvious, that, when Achilles overtakes the tortoise, their travels must have been as 100 to 1, and that the travel of Achilles is 1 mile increased by the travel of the tortoise. Consequently, the travel of the tortoise increased by 1 mile is to the same travel as 100 to 1. The present question therefore is, What is the number, which, increased by 1, is to itself as 100 to 1? This is the same as Question II. Hence, by its general Rule,

$$100-1=99, \text{ and } 99 : 1 :: 1 : \frac{1 \times 1}{99}=\frac{1}{99}.$$

Achilles will then overtake the tortoise at a distance of a mile and $\frac{1}{99}$ of a mile, where we find $1\frac{1}{99} : \frac{1}{99} :: 100 : 1$, or $\frac{100}{99} : \frac{1}{99} :: 100 : 1$, as ought to be.

Quest. V. If 180*l.* is to be distributed among two persons A and B, in such sort, that as often as A take 5, B shall take 4; what will be the share of each of them?

Because the share, for instance, of A must be 180 pounds minus the share of B by the first condition, it is evident, that 180 pounds minus the share of B is to the same share, as 5 to 4, by the last condition. The question being therefore of the same nature, as the third, the answer will be as follows,

$5+4=9,\ 9 : 4 :: 180 : \frac{180 \times 4}{9}=\frac{20 \times 4}{1}=80$, B's share. Hence $180-80=100$, A's share, and $100 : 80 :: 5 : 4$, as ought to be.

CHAPTER IX.

THE RULE OF INDUCTION.

THE Rule of Induction considers several examples, in order to give a general answer to the question proposed.

Rule of Induction. *Find any Rule for a particular case, and applying it suitably to some others which are similar, you will easily perceive whether this Rule may be taken as a general one.*

Quest. I. The ratios between two, three, or more factors being given, to find these factors.

When the factors are two, the rule is given in the preceding Chapter.

Suppose now it is required to find three factors of 3000, which are as 2, 3, and 4. Divide the given number 3000 by the product 24 of the terms, which express the ratios of its factors, as directed by the same Rule; extract the cubic root, because the factors are three, from the quotient hence arising, viz. from 3000 ÷ 24, or 125; multiply severally each term 2, 3, 4, by this root, or by 5, and you will find the three products 10, 15, 20, which are as 2, 3, 4, and multiplied together, produce the very number 3000, whereof they are consequently the three factors required.

Again,

Again, let four factors of the number 2880 be as 2, 3, 5, 6; the following calculation will answer to the purpose.

$2880 \div 2 \times 3 \times 5 \times 6 = 2880 \div 180 = 16$; $\sqrt[4]{16} = 2$,
$2 \times 2 = 4$, $2 \times 3 = 6$, $2 \times 5 = 10$, $2 \times 6 = 12$,
$4 \times 6 \times 10 \times 12 = 2880$; and 4, 6, 10, 12, as 2, 3, 5, 6.

The same method will be found to succeed in a greater number of examples, and therefore we may establish this general

RULE. *Divide the given number by the product of all the terms which express the ratios of its factors, and extract from the quotient the root of that index, which has so many units as there are factors; then multiply severally each term of those ratios by this root, and the products will give the factors required.*

Example.

What are the five factors of 3240, which are as 2, 3, 3, 5, 6?

Ans. They are $2\sqrt[5]{6}$, $3\sqrt[5]{6}$, $3\sqrt[5]{6}$, $5\sqrt[5]{6}$, $6\sqrt[5]{6}$; for $3240 \div 2 \times 3 \times 3 \times 5 \times 6 = 3240 \div 540 = 162 \div 27 = 6$.

Quest. II. The ratios between the parts of a number being given, to find these parts.

First, I take the number 60, the parts of which, 36 and 24, are as 3 and 2. Now, because $36 = 3 \times 12$ and $24 = 2 \times 12$, we have $60 = 36 + 24 = 3 \times 12 + 2 \times 12 = \overline{3+2} \times 12 = 5 \times 12$, that is, $60 = 5 \times 12$, and con-

sequently $60 \div 5 = 12$. Therefore, if you divide the given number, 60, by the sum, 5, of the terms, 3 and 2, of the proposed ratio, and multiply the quotient hence arising by each term, 3 and 2, severally; the two products, 36 and 24, will give the parts required.

Secondly, The parts 30, 18, and 12, of the same number 60, are as 5, 3, and 2; therefore, being $30 = 5 \times 6$, $18 = 3 \times 6$, and $12 = 2 \times 6$, we find $60 = 5 \times 6 + 3 \times 6 + 2 \times 6 = \overline{5+3+2} \times 6 = 10 \times 6$, viz. $60 = 10 \times 6$, and consequently $60 \div 10 = 6$. Hence, it is obvious, that the preceding rule will also serve in this case. And, because considering a greater number of examples, we shall meet with the same consequences, we may establish the following general

RULE. *Divide the given number by the sum of the terms, which express the ratios of its parts; then multiply severally the quotient by each of those terms; and the products will give the parts required.*

Example.

I demand the parts of the same number 60, which are as 6, 5, 4, 3, 2.

Ans. The parts are 18, 15, 12, 9, and 6; because $6+5+4+3+2=20$, and $60 \div 20 = 3$; hence $6 \times 3 = 18$, $5 \times 3 = 15$, $4 \times 3 = 12$, $3 \times 3 = 9$, $2 \times 3 = 6$.

Quest. III. To find any number of means arithmetically proportional between two given numbers.

Let it be the natural progression 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. It is evident, that the number 2 is a mean arithmetical proportional between 1 and 3, the numbers 2 and 3 are two mean arithmetical proportionals between 1 and 4, the numbers 2, 3, and 4, are three

mean arithmetical proportionals between 1 and 5, and ſo on, becauſe the natural numbers increaſe by the ſame common difference $=1$. Hence it appears that all the mean proportionals will be known, as ſoon as the firſt is known by the continual addition of the common difference; and alſo, that the firſt mean proportional will be known as ſoon as the common difference be known, becauſe adding it to the firſt term of progreſſion, the ſum will be the ſaid mean proportional. Now, becauſe the common difference in natural progreſſion is 1, I muſt try how unity ariſes from two numbers whatſoever, for inſtance, 2 and 4, 2 and 5, 2 and 6, 3 and 8, 3 and 9, &c. I therefore obſerve, that

$\frac{4-2}{2}=2\div2=1, \frac{5-2}{3}=3\div3=1, \frac{6-2}{4}=4\div4=1,$

$\frac{8-3}{5}=5\div5=1, \frac{9-3}{6}=6\div6=1,$ &c. and that the numerator of each of theſe fractions is the difference of the given numbers, and the denominator unity increaſed by the number of mean proportionals between the ſaid two numbers. Wherefore I conclude by Induction, that a like fraction will in every caſe give the common difference of the mean arithmetical proportionals, and conſequently eſtabliſh the following general

RULE. *Subtract the leſs of the given numbers from the greater, and divide the difference by unity increaſed by the number of means required; the quotient will be the common difference, which added to the leſs given number will give the firſt mean, added to the firſt mean will give the ſecond, added to the ſecond will give the third, &c.*

Example.

I demand five means arithmetically proportional between 2 and 20.

Anſ.

Anf. $\frac{20-2}{1+5}=18\div6=3$, the common difference; hence the means required are 5, 8, 11, 14, 17, and the arithmetical progreffion is confequently 2, 5, 8, 11, 14, 17, 20.

Queft. IV. To find any number of means geometrically proportional, between two given numbers.

Taking the geometrical progreffion 1, 2, 4, 8, 16, 32, 64, 128, &c. and reafoning as in the preceding queftion, all the matter will be to find the common ratio, from which the mean proportionals may be made by continual multiplication. I therefore take for inftance, 4 and 16, 4 and 32, 4 and 64, 4 and 128, &c. and obferve, that $16\div4=4$ and $\sqrt{4}=2$, the common ratio; $32\div4=8$ and $\sqrt[3]{8}=2$, the common ratio; $64\div4=16$ and $\sqrt[4]{16}=2$, the common ratio; $128\div4=32$ and $\sqrt[5]{32}=2$, the common ratio, &c. Hence the index of every root being unity increafed by the number of means required, we have the following general

RULE. *Divide the greater of the two given numbers by the lefs, and extract from the quotient that root, the index of which is unity increafed by the number of means required, fo fhall the refult be the common ratio, by which multiplying the lefs number, you will have the firft mean proportional, multiplying the firft mean proportional, you will have the fecond, multiplying the fecond, you will have the third, &c.*

Example.

Let it be required to find three mean proportionals between 2 and 162.

Anf.

Anſ. $162 \div 2 = 81$ and $\sqrt[4]{81} = 3$, becauſe the means being 3, the index of the root ought to be $3+1=4$: hence the means required are $2 \times 3 = 6$, $6 \times 3 = 18$, $18 \times 3 = 54$, and the progreſſion 2, 6, 18, 54, 162.

CHAPTER X.

THE RULE OF RETROGRADATION.

THE Rule of Retrogradation teacheth how to find the number ſought, aſcending from the reſult given in the queſtion propoſed to the ſame number ſought.

Rule of Retrogradation. *Write arithmetically the queſtion propoſed, then take the reſult and retrograding, do the contrary operations to thoſe you find in the way; your laſt reſult will be the number ſought.*

Queſt. I. What number is that, which multiplied by 20, and divided by 6, gives 140 in the quotient?

Queſtion.	Retrogradation.	Verification.
Operations. $\left\{\begin{array}{l}\times 20\\ \div 6\end{array}\right.$	Reſult 140	42
	6	20
	—— $\times$	—— $\times$
Reſult 140	840	840
	20	6
	—— $\div$	—— $\div$
	Anſ. 42	140

Queſt.

Queſt. II. What number is that from which if you take $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder add $\frac{7}{16}$ of $\frac{1}{20}$, the ſum will be 10?

Queſtion.

Operations.
$$\begin{cases} -\frac{2}{7} \text{ of } \frac{3}{8} = \frac{-3}{28} \\ +\frac{7}{16} \text{ of } \frac{1}{20} = \frac{+7}{320} \end{cases}$$

Reſult 10

Retrogradation.

Reſult 10

$$\begin{array}{rl} & 10 \\ & 7 \div 320 \\ \hline & 10 - 7 \div 320 & - \\ & 3 \div 28 \\ \hline & & + \end{array}$$

Anſ. $10 + \frac{3}{28} - \frac{7}{320} = 10\frac{191}{2240}$

Verification.

$$\begin{array}{rl} 10 + \frac{3}{28} - \frac{7}{320} \\ -\frac{3}{28} + \frac{7}{320} \\ \hline 10 & + \end{array}$$

Queſt. III. A ſtealing apples was taken by B, and to appeaſe him, gives him half what he had, and B gives him back 10; and going further met with C, and was forced to give him half of what he had left, and he returns him back 4; and going further meets D, and gives him half what he then had, and he returns him back 1; and getting ſafe away, finds he had 13 left: what had he at firſt?

Queſtion.

Question		Retrogradation.		Verification.	
		Result 13		60	
Operations.	÷ 2	1		—	÷ 2
	+ 10	—	−	30	
	÷ 2	12		10	
	+ 4	—	× 2	—	+
	÷ 2	24		40	
	+ 1	4		—	÷ 2
Result	13	—	−	20	
		20		4	
		—	× 2	—	+
		40		24	
		10		—	÷ 2
		—	−	12	
		30		1	
		—	× 2	—	+
		Ans. 60		13	

Quest. IV. A tradesman increased his stock annually by 100*l.* more than ¼ part of it, and at the end of 4 years found that it amounted to 10342*l.* 3*s.* 9*d.* what had he at setting out?

Question.

Operations.	÷ 4
	× 5
	+ 100
	÷ 4
	× 5
	+ 100
	÷ 4
	× 5
	+ 100
	÷ 4
	× 5
	+ 100
Result *l.*	10342 3 9

Note. Dividing any number by 4, then multiplying the quotient by 5, you will have the same result that arises from adding to the said number its fourth part.

Retrogradation.

	l.	s.	d.	
Refult	10342	3	9	
	100			−
	10242	3	9	÷5
	2048	8	9	×4
	8193	15		
	100			−
	8093	15		÷5
	1618	15		×4
	6475			
	100			−
	6375			÷5
	1275			×4
	5100			
	100			−
	5000			÷5
	1000			×4
	4000			

Verification.

l.	s.	d.	
4000			÷4
1000			×5
5000			
100			+
5100			÷4
1275			×5
6375			
100			+
6475			÷4
1618	15		×5
8093	15		
100			+
8193	15		÷4
2048	8	9	×5
10242	3	9	
100			+
10342	3	9	

CHAP.

CHAPTER XI.

THE RULE OF A NEW UNKNOWN QUANTITY.

THE Rule of a new Unknown Quantity finds the numbers sought by determining their sum, difference, product, or any other relation.

Rule of a new Unknown Quantity. *Find the sum, or any other relation of the numbers required; then determine those numbers according to the conditions of the question proposed.*

Quest. I. Three men had each a certain sum of money; A and B together had 16*l.* B and C together had 27*l.* A and C together had 25*l.* how much had each?

Let us determine the sum of all the three men. Now I observe, that by the conditions of the question $16+27+25$, or 68, contains twice this sum, which consequently ought to be $68\div 2$, that is, 34. Hence, because A and B together had 16*l.* C therefore had $34-16$, viz. 18*l.* which subtracted from 27*l.* gives the remainder 9*l.* for the money of B, and subtracted from 25*l.* gives the remainder 7*l.* for the money of A.

This solution written arithmetically is as follows:

A and B = 16
B and C = 27
A and C = 25
—— +
Twice the ſum = 68
—— ÷ 2
Sum required = 34

A, B, and C = 34
A and B = 16
—— –
C's money = 18

B and C = 27
C = 18
—— –
B's money = 9

A and C = 25
C = 18
—— –
A's money = 7

Queſt. II. I demand four numbers, which anſwer to the following conditions, viz.

that the ſum of the 1ſt, 2d, and 3d = 120
1ſt, 2d, and 4th = 140
1ſt, 3d, and 4th = 160
2d, 3d, and 4th = 180
—— +
Anſ. Three times the ſum of the numbers = 600
—— ÷ 3
The ſum required = 200

Hence the 1ſt number = 200 – 180 = 20,
the 2d = 200 – 160 = 40,
the 3d = 200 – 140 = 60,
the 4th = 200 – 120 = 80.

Queſt. III. I require four numbers anſwering to hefe conditions; viz.

that

that the ſum of the 1ſt and 2d = 60
1ſt and 3d = 80
1ſt and 4th = 100
2d and 3d = 100
2d and 4th = 120
3d and 4th = 140

Anſ. Three times the ſum of the numbers = 600

The ſum required = 200

Or, 1ſt and 2d = 60
3d and 4th = 140

The ſum required = 200

1ſt and 2d = 60
1ſt and 3d = 80
1ſt and 4th = 100

Their total = 240
Sum = 200

Twice the 1ſt number = 40

The 1ſt number = 20

1ſt and 2d = 60
1ſt = 20

2d = 40

1ſt and 3d = 80
1ſt = 20

3d = 60

1ſt and 4th = 100
1ſt = 20

4th = 80

Note. This queſtion offers two ſuperfluous conditions, which being not inconſiſtent with the others, do not oppoſe its poſſibility.

Queſt.

Quest. IV. I demand three numbers with the following conditions, viz.

that the product of the 1st and 2d $=150$
1st and 3d $=180$
2d and 3d $=270$

Anf. The product of their squares $=150\times180\times270$
$=7290000$.

Hence the product of the numbers $=\sqrt{7290000}=2700$;
the 1st $=2700\div270=10$;
the 2d $=2700\div180=15$;
the 3d $=2700\div150=18$.

Quest. V. To find four numbers anfwering to thefe conditions, viz.

that the product of the 1st, 2d, and 3d $=27000$
1st, 2d, and 4th $=30000$
1st, 3d, and 4th $=90000$
2d, 3d, and 4th $=270000$

Anf. The product of their cubes $=2700000\times2700000\times2700000$.
Hence the product of the numbers $=$
$\sqrt[3]{2700000\times2700000\times2700000}=2700000$.
The 1st $=2700000\div270000=10$;
the 2d $=2700000\div90000=30$;
the 3d $=2700000\div30000=90$;
the 4th $=2700000\div27000=100$.

Quest.

Queſt. VI. I demand four numbers, which anſwer to theſe conditions, viz.

that the product of the 1ſt and 2d $= 300$
1ſt and 3d $= 900$
1ſt and 4th $= 1000$
2d and 3d $= 2700$
2d and 4th $= 3000$
3d and 4th $= 9000$

Anſ. $1\text{ſt} \times 2\text{d} = 300$
$3\text{d} \times 4\text{th} = 9000$
$\times$

Product of all the numbers $= 2700000$.

Hence being $1\text{ſt} \times 2\text{d} = 300$
$1\text{ſt} \times 3\text{d} = 900$
$1\text{ſt} \times 4\text{th} = 1000$
$\times$

Product of all the numbers multiplied by the 1ſt ſquared $\} 270000000$

It will be 270000000
2700000
$\div$

the 1ſt number ſquared $= 100$
$\surd$

the 1ſt number $= 10$; therefore

The 2d $= 300 \div 10 = 30$;
the 3d $= 900 \div 10 = 90$;
the 4th $= 1000 \div 10 = 100$.

Note. There are two unneceſſary conditions, but not inconſiſtent with the others.

CHAP.

CHAPTER XII.

THE RULE OF PARTITION.

THE Rule of Partition is a method of parting a given ſum among any number of perſons, according to their different ratios.

Rule of Partition. *Determine every greater part from its ratio to the leſs; take the ſum of all the parts ſo determined; divide by it the given ſum, and the quotient will give the leſs part, from which the other parts may be made out by multiplication.*

Queſt. I. A man died and left a legacy of 900*l.* to be diſpoſed of among four of his relations, viz. A, B, C, and D; which legacy is to be diſpoſed of in this order: B is to have twice as much as A, and C thrice as much as B, and D is to have as much and half as much as C; what muſt each perſon have?

Anſ.	A muſt have	1 part.
	B	2 parts.
	C	6 parts.
	D	9 parts.
		— +
	All together	18 parts.

The

The legacy = 900
The ſum = 18

—— ÷

A's ſhare	= 50	= 50
B's	= 50 × 2	= 100
C's	= 50 × 6	= 300
D's	= 50 × 9	= 450

—— +

Total = 900

Queſt. II. Twenty knights, 30 merchants, 24 lawyers, and 24 citizens, ſpent at a dinner 64 pounds, which ſum was divided among them in ſuch manner, that 4 knights paid as much as 5 merchants, 10 merchants as much as 16 lawyers, and 8 lawyers as much as 12 citizens; the queſtion is, to know the ſum of money paid by all the knights, alſo by the merchants, lawyers, and citizens.

Anſ.

$$\overset{kn.}{4} : \overset{m.}{5} :: \overset{kn.}{20} : \overset{m.}{25}.$$

$$\overset{m.}{10} : \overset{l.}{16} :: \begin{cases} \overset{m.}{25} : \overset{l.}{40}. \\ 30 : 48. \end{cases}$$

$$\overset{l.}{8} : \overset{c.}{12} :: \left.\begin{cases} \overset{l.}{40} : \overset{c.}{60} \\ 48 : 72 \\ 24 : 36 \end{cases} \atop \text{Add } 24 \right\} \text{partial ſhares,}$$

—— +

Total 192.

Hence $64 \div 192 = \frac{1}{3}$.

Therefore the 20 knights paid	$60 \times \frac{1}{3} = 20$ pounds.	
30 merchants	$72 \times \frac{1}{3} = 24$	
24 lawyers	$36 \times \frac{1}{3} = 12$	
24 citizens	$24 \times \frac{1}{3} = 8$	

—— +

Sum 64.

Quest. III. A person dying, left his wife with child, and making his will, ordered that if she went with a son, $\frac{2}{3}$ of his estate, which was 6300*l.* should belong to him, and the remainder to his mother; and if she went with a daughter, he appointed the mother $\frac{2}{3}$, and the girl the remainder; but it happened that she was delivered both of a son and a daughter: what ought to be in this case the share of each?

Ans. It is evident, that by the father's will the son must have twice as much as the mother, and the mother twice as much as the daughter. Therefore, their shares must be as 4, 2, and 1. Hence

$$6300 \div 4+2+1 = 6300 \div 7 = 900 \left\{ \begin{array}{l} \text{for the daughter's} \\ \text{share.} \end{array} \right.$$
$$900 \times 2 = 1800 \text{ mother's.}$$
$$900 \times 4 = 3600 \text{ son's.}$$
$$\text{Sum} = 6300$$

CHAPTER XIII.

THE RULE OF FRACTIONS.

WHEN only a part of a whole is given in numbers, and the others are expressed by their relations to the same whole, *the Rule of Fractions* shews how to find them.

Rule of Fractions. *Reduce all the given fractions to the common denominator, add them together, and subtract the*

the ſum from unity; the remainder will ſhew what part of the whole is the given number; and hence, by the Rule of Three, all the parts will be found.

Queſt. I. The account of a certain ſchool is as follows, viz. $\frac{1}{16}$ of the boys learn geometry, $\frac{3}{8}$ learn grammar, $\frac{3}{10}$ learn arithmetic, $\frac{3}{20}$ learn to write, and 18 learn to read; I demand the number of each?

Anſ. $\frac{1}{16}+\frac{3}{8}+\frac{3}{10}+\frac{3}{20}=\frac{5}{80}+\frac{30}{80}+\frac{24}{80}+\frac{12}{80}=\frac{71}{80}$; then $1-\frac{71}{80} \times = \frac{9}{80}$. Hence the 18 readers are $\frac{9}{80}$ of the whole, and conſequently

$$9 : 18, \text{ or } 1 : 2 :: \begin{cases} 5 : 10 \text{ geometricians.} \\ 30 : 60 \text{ grammarians.} \\ 24 : 48 \text{ arithmeticians.} \\ 12 : 24 \text{ writers.} \end{cases}$$

Queſt. II. Three perſons purchaſe together a ſhip, toward the payment of which A advanced $\frac{2}{9}$, and B $\frac{2}{7}$ of the value, and C 200*l*, how much paid A and B, and what part of the veſſel had C?

Anſ. $\frac{2}{9}+\frac{2}{7}=\frac{14}{63}+\frac{18}{63}=\frac{32}{63}$; and $1-\frac{32}{63}=\frac{31}{63}$.
Hence $\frac{31}{63}$ is the part of the veſſel belonging to C for 200 pounds.

Therefore the payments of A and B are as follows,

$$31 : 200 :: \begin{cases} 14 : \dfrac{2800}{31} = 90\frac{10}{31} = 90l.\ 6s.\ 5d.\ 1\frac{21}{31} \text{ for A.} \\ 18 : \dfrac{3600}{31} = 116\frac{4}{31} = 116l.\ 2s.\ 6d.\ 3\frac{27}{31} \text{ for B.} \end{cases}$$

CHAPTER XIV.

THE RULE OF RELATIONS, WHEREIN OF ARBITRATION OF EXCHANGES.

WHEN the *coins, weights*, or *meaſures* of ſeveral countries are compared in the ſame queſtion, the *Rule of Relations* gives the anſwer required.

Note. This Rule is alſo called *Conjoined Proportion*.

Rule of Relations. 1. *Diſtinguiſh the given rates or prices into antecedents and conſequents; place the antecedents in one column, and the conſequents in another on the right, fronting one another by way of equation.*

2. *The ſecond antecedent muſt be of the ſame kind with the firſt conſequent, and the third antecedent of the ſame kind with the ſecond conſequent, &c. ſo that the laſt conſequent ſhall be of the ſame kind as the firſt antecedent.*

3. *If to any of the numbers a fraction be annexed, both the antecedent and the conſequent muſt be multiplied into the denominator.*

4. *Multiply the antecedents continually for a product, and the conſequents continually for another product.*

5. *Laſtly, divide the product ariſing from a greater number of factors by the other; and the quotient will be the anſwer required.*

Queſt.

Quest. I. If 3 dozen pair of gloves be equal in value to 2 pieces of ribbon; 3 pieces of ribbon to 7 dozen of points; 6 dozen of points to 2 yards of Flanders lace; and 3 yards of Flanders lace to 81 shillings; how many dozen pair of gloves may be bought for 28 shillings?

$$\begin{array}{rrcl} \text{Ans.} & 3g. & = & 2r. \\ & 3r. & = & 7p. \\ & 6p. & = & 2l. \\ & 3l. & = & 81sh. \\ & 28sh & = & ?\,g. \end{array}$$

Hence $\frac{3 \times 3 \times 6 \times 3 \times 28}{2 \times 7 \times 2 \times 81} = \frac{28}{2 \times 7} = \frac{28}{14} = 2$ dozen pair of gloves.

Quest. II. If *6lb.* of sugar be equal in value to *7lb.* of raisins, *5lb.* of raisins to *2lb.* of almonds, *3lb.* of almonds to *5lb.* of currants, *2lb.* of currants to *18d.* how many pence are the value of *3lb.* of sugar?

$$\begin{array}{rrcl} \text{Ans.} & 6s. & = & 7r. \\ & 5r. & = & 2a. \\ & 3a. & = & 5c. \\ & 2c. & = & 18d. \\ & ?\,d. & = & 3s. \end{array}$$

Hence $\frac{7 \times 2 \times 5 \times 18 \times 3}{6 \times 5 \times 3 \times 2} = 7 \times 3 = 21d.$

Note. It is obvious, that the Rule of Relations is nothing else, but a conjunction of some proportions; from whence it is called *Conjoined Proportion.*

A R-

ARBITRATION OF EXCHANGES.

THE courſe of exchange betwixt nation and nation naturally riſes or falls according as the circumſtances and balance of trade happens to vary. Now to draw upon, and remit to foreign places, in this fluctuating ſtate of exchange, in the way that will turn out moſt profitable, is the deſign of arbitration, which is either ſimple or compound.

I. *Simple Arbitration.*

In Simple Arbitration the rates or prices of exchange from one place to two others are given; whereby is found the correſpondent price between the ſaid two places, called the *arbitrated price*, or *par of arbitration*; and hence is derived a method of drawing and remitting to the beſt advantage.

Queſt. I. If exchange from London to Amſterdam be 33*s.* 9*d.* per *l.* ſterling; and if exchange from London to Paris be 32*d.* per crown; what muſt be the rate of exchange from Amſterdam to Paris, in order to be on a par with the other two?

Anf. Amft. $33s.\ 9d. = 1l. = 240d.$ Lond.
Lond. $32d. = 1cr.$ Par.
Par. $1cr. = ?d.$ Amft.

$$\text{Hence } \frac{33s.\ 9d. \times 32d. \times 1cr.}{240d. \times 1cr.}, \text{ or}$$

$$\frac{\overline{3 \times 11s. + 3 \times 3d.} \times 2 \times 16d. \times 1cr.}{3 \times 5 \times 16d. \times 1cr.} = \frac{\overline{11s. + 3d.} \times 2}{5} =$$

$\dfrac{22s.\ 6d.}{5} = 4s.\ 6d. = 54d.$ the rate of exchange required.

Queſt. II. If exchange from London to Paris be $32d.$ ſterling per crown, and to Amſterdam $405d.$ Flemiſh per $1l.$ ſterling; and if, by advice from Holland or France, the price of exchange between Paris and Amſterdam is fallen to $52d.$ Flemiſh per crown; what may be gained per cent. by drawing on Paris, and remitting to Amſterdam?

Anſ. Direct draught to Paris.

Lond. $32d. = 1cr.$ Par.
Paris $1cr. = 24000d.$ Lond.

$$\text{Hence } \frac{24000}{32} = 750cr.$$

Circular remittance by way of Amſterdam.

Par. $1cr. = 52d.$ Amft.
Amft. $405d. = 1l.$ Lond.
Lond. $?\ l. = 750cr.$ Par.

$$\text{Hence } \frac{52 \times 750}{405} = \frac{52 \times 50}{27} = \frac{2600}{27} = 96\tfrac{8}{27}l.$$

Direct draught $= l.\ 100$
Circular remittance $l.\ 96\frac{8}{27} = l.\ 96\ \ 5\ \ 11\frac{1}{9}$

Gain per cent. $= l.\ 3\ \ 14\ \ \ \frac{8}{9}$

Queſt.

Quest. III. London is indebted to Petersburgh 5000 rubles, and Petersburgh can draw for them directly on England at 50 pence sterling per ruble, or on Holland at 90 pence Flemish per ruble; which of these two ways will be most advantageous to London, supposing the course of exchange between London and Holland to be 36*s.* 4*d.* Flemish per 1*l.* sterling?

Ans. The direct draught to Petersburgh.

Lond. 50*d.* = 1*r.* Pet.
Pet. 5000*r.* = ?*d.* Lond.

Hence $5000 \times 50 = 250000d.$ Lond.

The circular remittance by way of Holland.

Lond. 240*d.* = 436*d.* Hol.
Hol. 90*d.* = 1*r.* Pet.
Pet. 5000*r.* = ?*d.* Lond.

Hence $\frac{240 \times 90 \times 5000}{436} = 247706\frac{46}{109}d.$ Lond.

		l.	*s.*	*d.*
The direct draught	$250000d.$ =	1041	13	4
The circular remittance	$247706\frac{46}{109}d.$ =	1032	2	$2\frac{46}{109}$
	$2293\frac{63}{109}d.$	9	11	$1\frac{63}{109}$

II. *Com-*

II. *Compound Arbitration.*

IN Compound Arbitration the rate or price of exchange between three, four, or more places, is given, in order to find how much a remittance passing through them all will amount to at the last place; or to find the arbitrated price, or par of arbitration, between the first place and the last.

Quest. If London remit 1000*l.* to Spain, by way of Holland, at 35*s.* Flemish per 1*l.* sterling, thence to France, at 58*d.* Flemish per crowns; thence to Venice, at 100 crowns per 60 ducats; and thence to Spain, at 360 mervadies per ducat; how many piastres, of 272 mervadies each, will the 1000*l.* amount to in Spain?

Ans. 1*l.* sterling = 420*d.* Flem.
58*d.* Flem. = 1*cr.* France.
100*cr.* Fr. = 60*ducats* Venice.
1*duc.* Ven. = 360*merv.* Spain.
272*merv.* Sp. = 1*piastre.*
?*piastres.* = 1000*l.* sterling.

$$\text{Hence } \frac{420 \times 1 \times 60 \times 360 \times 1 \times 1000}{1 \times 58 \times 100 \times 1 \times 272}, \text{ or}$$

$$\frac{210 \times 15 \times 90 \times 10}{29 \times 17} = \frac{2835000}{493}, \text{ that is, } 5750\tfrac{250}{493} \text{ piastres.}$$

Quest. II. A banker in Amsterdam remits to London 400*l.* Flemish; first to France, at 56*d.* Flemish per crown; from France to Venice, at 100 crowns per 60 ducats; from Venice to Hamburgh, at 100*d.* Flemish per ducat;

cat; from Hamburgh to Lisbon, at 50*d.* Flemish per crusade of 400 rees; and lastly, from Lisbon to London, at 64*d.* sterling per milree or 1000 rees: how much sterling money will the remittance amount to? and how much will be gained or saved, supposing the direct exchange from Holland to London at 442*d.* Flemish per 1*l.* sterling?

The direct draught to London.

$$\begin{array}{lcl} 442d. \text{ Flem.} & = & 1l. \text{ sterling.} \\ ?\ l. \text{ sterl.} & = & 96000d. \text{ Flem.} \end{array}$$

$$\text{Hence } \frac{96000 \times 1}{442} = \frac{48000}{221} = 217\tfrac{43}{221}l. \text{ ster.}$$

The circular remittance with London.

$$\begin{array}{lcl} 56d. \text{ Flem.} & = & 1cr. \\ 100cr. & = & 60duc. \\ 1duc. & = & 100d. \text{ Flem.} \\ 50d. \text{ Flem.} & = & 400rees. \\ 1000rees. & = & 64d. \text{ sterl.} \\ ?\ d. \text{ sterl.} & = & 96000d. \text{ Flem.} \end{array}$$

$$\text{Hence } \frac{1 \times 60 \times 100 \times 400 \times 64 \times 96000}{56 \times 100 \times 1 \times 50 \times 1000}, \text{ or}$$

$$\frac{60 \times 8 \times 8 \times 96}{7} = \frac{368640}{7} = 52662\tfrac{6}{7}d. \text{ sterl.}$$

	l.	*s.*	*d.*
Circular remittance $52662\frac{6}{7}d.$ sterl. =	219	8	$6\frac{6}{7}$
Direct draught $217\frac{43}{221}l.$ sterl. =	217	3	$10\frac{154}{221}$
Gained by the circular exchange	2	4	$8\frac{248}{1547}$

Quest.

Quest. III. A merchant at London has credit for 680 piastres at Leghorn, for which he can draw directly at 50*d.* sterling per piastre; but chusing to try the circular way, they are by his order remitted, first to Venice, at 94 piastres per 100 ducats banco; thence to Cadiz, at 320 mervadies per ducat; thence to Lisbon, at 630 rees per piastre of 272 mervadies; thence to Amsterdam, at 50*d.* Flemish per crusade of 400 rees; thence to Paris, at 56*d.* Flemish per crown; thence to London, at $31\frac{1}{3}d.$ sterl. per crown, or at 94*d.* sterling per 3 crowns: what is the arbitrated price between London and Leghorn per piastre? and how much is the circular remittance better than the direct draught, without reckoning charges?

The arbitrated price between London and Leghorn.

94 *piast.*	=	100 *duc.*
1 *duc.*	=	320 *merv.*
272 *merv.*	=	630 *rees.*
400 *rees.*	=	50*d.* Flem.
56*d.* Fl.	=	1 *cr.*
3 *cr.*	=	94*d.* sterl.
? *d.* sterl.	=	1 *piastre.*

$$\text{Hence } \frac{100 \times 320 \times 630 \times 50 \times 1 \times 94 \times 1}{94 \times 1 \times 272 \times 400 \times 56 \times 3}, \text{ or}$$

$$\frac{5 \times 15 \times 25}{34} = \frac{1875}{34} = 55\tfrac{5}{34}d. \text{ sterl.}$$

		l.	*s.*	*d.*
Direct draught to Leghorn, 680 piastres, at 50*d.* sterl.	=	141	13	4
Circular remittance, at $55\frac{5}{34}d.$ sterl.	=	156	5	0
	Gained	14	11	8

 In

In this circular exchange the agents or factors at the different places through which the money passes, retain so much in name of commission; and the regular accurate method of computation is, to deduce the commission from the several consequents.

Thus, if we resume the former question, and suppose the commission to be ½ per cent. the antecedents and consequents, with commission deduced, will stand as under.

94 *piastres*	=	99.5 *ducats*, commission deduced.
1 *duc.*	=	318.4 *merv.* commission deduced.
272 *merv.*	=	626.85 *rees*, commission deduced.
400 *rees*	=	49.75*d.* Flem. commission deduced.
56*d.* Fl.	=	.995 *cr.* commission deduced.
3 *cr.*	=	94*d.* sterl.
? *d.* st.	=	1 *piastre.*

By working as formerly the arbitrated price or value of the piastre will be found to be 53.78*d.* sterling nearly, which is something less than the answer found, exclusive of charges. But still there will be profit by the circular remittance; for,

		l.	*s.*	*d.*
The circular remittance, 680 piastres, at 53.78*d.* sterl.	=	152	7	6
The direct draught, 680 piastres, at 50*d.* sterl.	=	141	13	4
	Gain =	10	14	2

The value of the above piastre, with commission allowed, may be found easily, and pretty nearly, by deducing five commissions, equal to 2½ per cent. or $\frac{1}{40}$ from the arbitrated price, exclusive of charges, thus:

The

The arbitrated price $55\frac{5}{34}$ = 55.147

The value of five commissions $\frac{1}{40} \times 55\frac{5}{34}$ } = 1.378

53.769 value nearly.

Note. A person who knows at what rate he can draw or remit directly, and at the same time has advice of the course of exchange in foreign places, may chalk out a path for circulating his money, so as to make a benefit of his skill and credit: and herein lies the great art of such negociations.

CHAPTER XV.

THE RULE OF CORRECTIONS.

THE Rule of Corrections is so called, because by the help of *positions* or false supposed numbers, a *correction* is found, which added to the first position gives the true number required.

Rule of Corrections. 1. *Take two positions and work with each of them as if it was the true number, till by this means you bring out the results, and consequently the errors; then,*

2. *Multiply the first error by the excess of the second position above the first.*

3. *Divide the product by the excess of the first error above the second.*

4. *Lastly, add to the quotient the first position; the result will be the number sought.*

Quest.

Quest. I. A, B, and C, would divide 200*l.* between them so that B may have 6*l.* more than A, and C 8*l.* more than B; how much must each man have?

1st position.		2d position.		True position.	
A	10	A	12	A	60
B	16	B	18	B	66
C	24	C	26	C	74
	—— +		—— +		—— +
Total	50	Total	56		*l.*200
	*l.*200		*l.*200		
	——		——		
1st er. = 150		2d error = 144			

Excess of the 2d position above the 1st, $12-10=2$.
Excess of the 1st error above the 2d, $150-144=6$.

The number sought $=\frac{150\times 2}{6}+10=50+10=60.$

Quest. II. A schoolmaster being asked how many scholars he had, said, if I had as many, half as many, and ¼ as many more, I should have 88; how many had he?

	1st position.	2d position.	True position.
Suppose he had	40	44	32
Then as many	40	44	32
½	20	22	16
¼	10	11	8
	—— +	—— +	—— +
Total	110	121	Scholars 88
Scholars	88	88	
	—— −	—— −	
	1st error = 22	2d error = 33	

Excess of the 2d position above the 1st, $44-40=4$.
Excess of the 1st error above the 2d, $110-121=-11$.

The number sought $=\frac{22\times 4}{-11}+40=\frac{2\times 4}{-1}+40=$
$-8+40=32.$

Note. We supposed 40 and 44 for the number sought, in order to avoid fractions. It is obvious, that the correction −8 in the second question is negative or to be subtracted, because the first position is greater than the number required.

CHAPTER XVI.

THE RULE OF CONCOURSE.

WHEN an unknown number is affected by two conditions, it is by the *Rule of Concourse* that it may be determined.

Rule of Concourse. *If an unknown number is affected by two conditions, give to it two different values in each position, in order to satisfy both the conditions; then work the differences between those values in the same manner, as the errors in the Rule of Corrections; the result brought out by the same operation is the number sought.*

Note. This Rule is called of *Concourse*, because the two different values given to the unknown number in each false position, concur to be one and the same in the true position.

Quest. I. One has 6 sons, each whereof is 4 years older than his next youngest brother, the eldest is 3 times as old as his youngest brother; what are their several ages?

Ans.

Anf.	1ſt poſition.	2d poſition.	True poſition.
Age of youngeſt ſon =	3	5	10
5th	7	9	14
4th	11	13	18
3d	15	17	22
2d	19	21	26
1ſt	23	25	30
1ſt	9	15	30
	1ſt difference 14	2d diff. 10	0

Exceſs of the 2d poſition above the firſt $= 5 - 3 = 2$.
Exceſs of the 1ſt difference above the 2d $= 14 - 10 = 4$.

The number ſought $= \frac{14 \times 2}{4} + 3 = 7 + 3 = 10$.

Queſt. II. Three perſons A, B, C, owe a certain ſum of money, ſo that A and B together owe 210 crowns, B and C 290, and C and A 400; what did each of them owe?

Anſ.	1ſt poſition.	2d poſition.	True poſition.
A	100	110	160
B	110	100	50
C	180	190	240
C	300	290	240
	1ſt difference 120	2d diff. 100	0

Exceſs of the 2d poſition above the 1ſt $= 110 - 100 = 10$.
Exceſs of the 1ſt diff. above the 2d $= 120 - 100 = 20$.

The number ſought $= \frac{120 \times 10}{20} + 100 = 60 + 100 = 160$.

CHAP.

CHAPTER XVII.

THE RULE OF TRANSFORMATIONS,

WHEREIN OF THE QUESTIONS WITH MANY UNKNOWN QUANTITIES.

WHEN the conditions of the queſtion propoſed are not plain enough to draw the values of the different unknown numbers from the arbitrary value given to one of them, the *Rule of Transformations* ſhews how to change the queſtion in order to facilitate the arbitrary determination of the unknown numbers.

Note. This Rule is very uſeful in every queſtion, becauſe by this way the anſwer will be eaſily found, as will appear by examples.

Definition. The number which ſhews how many times an unknown quantity is taken in a condition of the queſtion propoſed, is called its *coefficient.*

Rule of Transformation. *Write arithmetically the conditions of the queſtion; take away the fractions from every condition, if need be, multiplying it by the leaſt multiple of all the denominators; then, chuſing any unknown quantity you pleaſe, multiply each coefficient, which it hath in one condition, by all the coefficients it hath in the others, in order to give to the ſaid quantity the ſame coefficient in every condition; and laſtly, either by Addition or Subtraction change the conditions to have new ones, all wanting*

that unknown quantity. Performing the same operations upon the new conditions, you may diminish the number of the unknown quantities, till you get a condition with only one, which will be very easily determined.

Note. The number of the new conditions must be one less than the former.

Quest. I. Two men have a mind to purchase a house rated at 1200 pounds; says A to B, if you give me $\frac{2}{3}$ of your money, I can purchase the house alone; but says B to A, if you will give me $\frac{3}{4}$ of yours, I shall be able to purchase the house; how much money had each of them?

Ans. 1st condition. A's money + $\frac{2}{3}$ of B's = 1200*l.*
2d condition. B's money + $\frac{3}{4}$ of A's = 1200*l.*

Multiplying the first condition by 3 and the second by 4, we shall have

3 times A's money + 2 times B's money = 3600*l.*
3 times A's money + 4 times B's money = 4800*l.*

* 2 times B's money = 1200*l.*

Hence B's money = 1200 ÷ 2 = 600, $\frac{2}{3}$ of which = 400;
A's money = 1200 − 400 = 800.

Quest. II. Three merchants met at an inn, and find the sum of their gains 780*l.* if you add the gain of the 1st and 2d, and from the sum subtract the gain of the 3d, there remains the gain of the 1st + 82 pounds; but if you add the gains of the 2d and 3d, and from the sum subtract the gain of the 1st, there remains the gain of the 3d − 43 pounds; what is the gain of each?

Ans.

Anſ. 1ſt condition. All the gains together=780 pounds.
2d condition. The gain of the 1ſt and 2d – the gain of the 3d=the gain of the 1ſt+82.
3d condition. The gain of the 2d and 3d – the gain of the 1ſt=the gain of the 3d – 43.

Clearing the 2d and 3d conditions, they are as follows:

2d condition.	The gain of the 2d – the gain of the 3d	$\}= 82l.$
3d condition.	The gain of the 2d – the gain of the 1ſt	$\}= -43l.$
1ſt condition.	The gain of the 2d, 3d, and 1ſt	$\}= 780l.$
		+
	3 times the gain of the 2d	$= 819l.$

Hence the gain of the 2d $=\frac{819}{3}=273l.$

The gain of the 3d=273 – 82=191*l.* by the 2d condition.

The gain of the 1ſt=273+43=316*l.* by the 3d condition.

+
780*l.*

Queſt. III. Three men have each ſuch a ſum of money, that if the 1ſt and 2d man's money be added to ½ of what the 3d man has, that ſum will be 92*l.* and if the 2d and 3d man's money be added to ⅓ of the 1ſt man's money, that ſum will be 92*l.* laſtly, if ¼ of the 2d man's money be added to the 1ſt and 3d man's money, that ſum will alſo be 92*l.* what is each man's money?

Anſ.

Anf. 1ft cond. $\left.\begin{array}{r}\text{The 1ft and 2d man's money}\\ +\frac{1}{2}\text{ of the 3d man's money}\end{array}\right\}=92l.$

2d condition. $\left.\begin{array}{r}\text{The 2d and 3d man's money}\\ +\frac{1}{3}\text{ of the 1ft man's money}\end{array}\right\}=92l.$

3d condition. $\left.\begin{array}{r}\text{The 1ft and 3d man's money}\\ +\frac{1}{4}\text{ of the 2d man's money}\end{array}\right\}=92l.$

Multiplying the 1ft condition by 2, the 2d by 3, the 3d by 4, they will be as follow:

A. $\left.\begin{array}{l}\text{2 times the 1ft and 2d man's money}+\\ \text{the 3d man's money}\end{array}\right\}=184l.$

B. $\left.\begin{array}{l}\text{3 times the 2d and 3d man's money}+\\ \text{the 1ft man's money}\end{array}\right\}=276l.$

C. $\left.\begin{array}{l}\text{4 times the 1ft and 3d man's money}+\\ \text{the 2d man's money}\end{array}\right\}=368l.$

Again, multiplying the new condition A by 2, and B by 4, it will be,

D. $\left.\begin{array}{l}\text{4 times the 1ft and 2d man's money}\\ +\text{2 times the 3d man's money}\end{array}\right\}=368l.$

E. $\left.\begin{array}{l}\text{4 times the 1ft man's money}+\text{12 times}\\ \text{the 2d and 3d man's money}\end{array}\right\}=1104l.$

F. $\left.\begin{array}{l}\text{8 times the 2d man's money}+\text{10 times}\\ \text{the 3d man's money}\end{array}\right\}=736l.$

Subtracting the condition C from E, it will remain,

G. $\left.\begin{array}{l}\text{11 times the 2d man's money}+\text{8 times}\\ \text{the 3d man's money}\end{array}\right\}=736l.$

Now, multiply the condition F by 11 and the condition G by 8, and you will find

$\left.\begin{array}{l}\text{88 times the 2d man's money}+\text{110}\\ \text{times the 3d man's money}\end{array}\right\}=8096l.$

$\left.\begin{array}{l}\text{88 times the 2d man's money}+\text{64}\\ \text{times the 3d man's money}\end{array}\right\}=5888l.$

---

46 times the 3d man's money $=2208l.$

Hence

Hence the 3d man's money $= \frac{2208}{46} = 48$;

and $10 \times 48 = 480$;

the 2d $= \frac{736 - 480}{8} = \frac{256}{8} = 32$, by condition F.

the 1st $= 92 - 32 - 24 = 36$, by the 1st condition.

CHAPTER XVIII.

THE RULE OF DIVISORS,

WHEREIN OF THE QUESTIONS OF EVERY DEGREE.

WHEN the number fought muft be an integer, the *Rule of Divifors* teacheth how to difcover it.

Note. This Rule is moft ufeful to refolve the queftions of every degree, viz. when the unknown quantities are multiplied together, or raifed to any power.

Rule of Divifors. *Find the divifors of the error arifing from a falfe pofition; then add to, or fubtract from, this pofition, as need requires, feverally each divifor, to correct the error; and among the refults or corrections will be the integer number fought. Now, to difcover it, take any correction you pleafe and work it as a new falfe pofition, in order to bring out the error, if the fuppofed correction does not anfwer to the conditions of the queftion; then find the divifors of the new error, and with them, either by Addition or by Subtraction, correct, as formerly, your*

your second position, to determine new corrections, among which will be also the integer number required, which consequently must be the number common to both the new and former corrections you had determined.

Notes. I. If the conditions of the question demand the false position to be divided by any number, you must multiply by the same number the divisors of the error, and then correct the false position with the products hence arising.

II. If you take a position with decimals, it will be enough to determine the number sought, as may be seen in the Examples.

Quest. I. A gentleman being asked how many horses he kept, made answer, for want of room in my own stable, I must put 8 horses in my neighbour's; but I am now building a stable twice as large, and then I can accommodate my own horses and 8 of my neighbour's; hence you may find out the number of horses I keep.

Ans.	1st position.	2d position.
Horses	20	22
	8	8
Stable's capacity	12	14
	× 2	× 2
Its double	24	28
All the horses	28	30
Error	= 4	2
Its divisors	1, 2, 4.	1, 2

Corrections $\begin{cases} 20+1=21 \\ 20+2=22 \\ 20+4=24^* \end{cases}$ $\quad 22+1=23$, $22+2=24^*$

Position with decimals.		True position.
	20.15	24
	8	8
	12.15	16
	$\times 2$	$\times 2$
	24.30	32
	28.15	32
Error	3.85	0
Position	20.15	
	+	
Correction	24.00	

Quest. II. One asked a shepherd how many sheep he had, and what was the price of his hundred sheep; I have not 100, said he, but if I had as many more and half as many more, and $7\frac{1}{2}$ sheep, then I should have just an hundred.

Ans.	1st position.	2d position.
Sheep	23	33
As many	23	33
$\frac{1}{2}$ as many	$11\frac{1}{2}$	$16\frac{1}{2}$
	$7\frac{1}{2}$	$7\frac{1}{2}$
	+	+
Total	65	90
	100	100
Error	35	10
Its divisors	1, 5, 7, 35	1, 2, 5, 10
	$\times 2$	$\times 2$
	2, 10, 14, 70	2, 4, 10, 20
Corrections	25, 33, 37, 93.	35, 37, 43, 53.
	*	*

Position

Position with decimals.	True position.
23.15	37
23.15	37
11.575	18½
7.5	7½
——— +	—— +
65.375	100
100	
— ———	

Error	34.625
Divisor	13.85
Position	23.15
	——— +
Correction	37.00

Quest. III. A gentleman having a certain number of greyhounds, said, if a third, a fourth, and a sixth of them were added together the sum would be 45; how many greyhounds had he?

Ans.	1st position.	2d position.
Greyhounds	48	84
	—	—
1÷3	16	28
1÷4	12	21
1÷6	8	14
	— +	— +
Total	36	63
	45	45
	— —	— —
Error	9	18
Divisors	1, 3, 9	1, 2, 3, 6, 18
	——— × 12	——— × 12
	12, 36, 108	12, 24, 36, 72, 216
Corrections	60, 84, 156	72, 60, 48, 12
	*	*

Position

Position with decimals.	True position.
48.15	60
16.05	20
12.0375	15
8.025 +	10 +
36.1125	45
45	

Error	8.8875
Divisor	11.85
Position	48.15 +
Correction	60.00

Notes. I. When the false position is to be divided by several numbers, the divisors of the error must be multiplied by the least multiple of all the said numbers.

II. The position with decimals does not want that multiplication, and its correction is also easily determined by the figures of decimals wanted to make the position an integer number. Thus, our position being with decimals, 48.15, the divisor must have necessarily the two last figures, 85, otherwise the number 48.15 could not be made an integer by Addition. If the same number 48.15 should be made an integer by Subtraction, the two last figures of the divisor should be 15.

Quest. IV. [Of the second degree.] Let 969 soldiers be drawn up into an oblong battle, so that the difference of the greater and less sides is 40: required the number of the soldiers of each rank in length and breadth?

Anſ.	1ſt poſition.	2d poſition.
The leſs ſide	19	18
The greater	59 ×	58 ×
The Battle	1121	1044
The ſoldiers	969 −	969 −
Error	152	75
Diviſors	1, 2, 4, 8, 19	1, 3, 5, 15
Corrections	18, 17*, 15, 11.	17*, 15, 13, 3.

Poſition with decimals.		True poſition.
19.15		17
59.15 ×		57 ×
1132.7225		969
969 −		
	Diviſor	2.15
163.7225	Poſition	19.15 −
5, 43, 215.	Correction	17.00

Notes. I. Although we find the numbers 15 and 17 common to both the corrections of the 1ſt and 2d poſitions, nevertheleſs the correction 17 only muſt be taken, becauſe the product of the two numbers ſought muſt end in 9.

II. The diviſors of the error 163.7225 are not ſo eaſy to be found, and therefore a table of diviſors of numbers like that publiſhed by M. Henry Anjema in 1767, at Leyden, would be very convenient.

Rule

Quest. V. [Of the third degree.] To find two numbers, whose sum is 20, and the product of the greatest multiplied by the square of the least is 768.

Anf.	1st position.	Position with decimals.
The least number	7	7.7
Its square	49	59.29
The greatest number	13 ×	12.3 ×
The product	637	729.267
	768	768
Error	131	38.733
Divisors	1, 131	1, 3
Correction	$7+1=8$	$7.7+.3=8.0$

True position.

8

64

12 ×

768

CHAPTER XIX.

THE RULE OF SERIES,

WHEREIN OF THE NATURE OF THE QUESTIONS OF ALL DEGREES.

Algebraical Series is a rank of numbers increaſing, or decreaſing, without interruption, according to the law, *that ſome differences, either 1ſt or 2d, 3d, &c. muſt be conſtant or the ſame*; and the ſeries is called of the *1ſt order*, if the 1ſt differences are conſtant, of the *2d order*, if the 2d differences are conſtant, and ſo on, as may be ſeen in the following Examples:

Algebraical ſeries of the 1ſt order.

1, 3, 5, 7, 9, 11, 13, &c.
2, 2, 2, 2, 2, 2, &c. 1ſt and conſtant differences.

Algebraical ſeries of the 2d order.

1, 4, 9, 16, 25, 36, 49, &c.
3, 5, 7, 9, 11, 13, &c. 1ſt differences.
2, 2, 2, 2, 2, &c. 2d and conſtant diff.

Algebraical ſeries of the 3d order.

1, 8, 27, 64, 125, 216, 343, &c.
7, 19, 37, 61, 91, 127, &c. 1ſt differences.
12, 18, 24, 30, 36, &c. 2d differences.
6, 6, 6, 6, &c. 3d and conſtant diff.

When

When the false positions are in an arithmetical progression, the errors thence arising form an algebraical series, which may be called the *resolving series*, because it gives the resolution of the question proposed.

The centre of the resolving series is the error answering to the position $=0$.

The root of the resolving series is a number answering to the conditions of the question.

The root of the resolving series is *rational*, when it is exactly expressed by a number, either integer, or fractional; but, if the root cannot be so expressed, it is *irrational*.

If two or more algebraical series be multiplied continually together, that is, the 1st term by the 1st, the 2d by the 2d, and so on, they will produce a new algebraical series, which may be called *compounded*, and the former *components*, as appears from the following Examples:

Example I.

1, 2, 3, 4, 5, 6, &c. 2, 4, 6, 8, 10, 12, &c.	Component series of the 1st order.
×	
2, 8, 18, 32, 50, 72, &c.	Compounded series of the 2d order.

Example II.

1,	2,	3,	4,	5,	6,	&c.	Component series of the 1st order.
2,	4,	6,	8,	10,	12,	&c.	
17,	14,	11,	8,	5,	2,	&c.	
×							
34,	112,	198,	256,	250,	144,	&c.	Compounded series of the 3d order.

Example.

Example III.

1, 2, 3, 4, 5, 6, &c.	Component ſeries of the 1ſt order.
1, 4, 9, 16, 25, 36, &c.	Component ſeries of the 2d order.
————————— ×	
1, 8, 27, 64, 125, 216, &c.	Compounded ſeries of the 3d order.

Principles. I. If the queſtion propoſed is of the 1ſt, 2d, 3d, &c. degree, the reſolving ſeries will be of the 1ſt, 2d, 3d, &c. order.

II. A reſolving ſeries has as many rational roots as component ſeries of the 1ſt order.

III. If a reſolving ſeries has no component of the 1ſt order, its roots are all irrational.

IV. To determine the reſolving ſeries of a queſtion propoſed, two poſitions are required if the queſtion is of the 1ſt degree, 3 poſitions if it is of the 2d, 4 if of the 3d, &c.

The Rule of Series is a method to find the reſolving ſeries and its roots.

Rule of Series. *Take from the natural progreſſion* 0, 1, 2, 3, 4, 5, *&c. as many poſitions as are required to determine the reſolving ſeries* (Princ. IV.); *then, if the ſeries is compounded, reſolve it into its components of the 1ſt order by finding the factors of its terms; and laſtly, divide the centre of each new ſeries by the conſtant difference, that is, by the exceſs of the ſame centre upon the next following term, and every quotient will be a root of the reſolving ſeries.*

Note.

Note. If the resolving series has no components of the 1st order, and therefore its roots are irrational (Pr. III.), they may be found by *approximation*, as we shall see in the next following Chapter XXI.

Quest. I. Two persons, A and B, travelling together, A with 100, and B with 48 guineas about him, met a company of robbers, who took twice as much from A as from B, and left A thrice as much as they left B; I demand how much they took from each?

Ans.	1st position.	2d position.	True position.
Taken from B,	0	1	44
Taken from A,	0	2	88
Left B,	48	47	4
	× 3	× 3	× 3
Left A,	144	141	12
Left A,	100	98	12
Error,	44	43	0

The resolving series of the 1st order } 44, 43, 42, 41, 40, 39, &c.
The constant differences 1, 1, 1, 1, 1, &c.
The centre of the series 44; its root $44 \div 1 = 44$.

Quest. II. Let there be a square whose side is 110 inches; it is required to assign the length and breadth of a rectangled parallelogram or long square, whose perimeter shall be greater than that of the square by four inches, but area shall be less than the area of the square by 4 square inches.

Ans. Since the side of the square is 110 inches, its area will be 12100 square inches; therefore the area of the parallelogram sought will be 12096 square inches: again,

again, the perimeter of the given ſquare is 440 inches; therefore the perimeter of the parallelogram ſought muſt be 444 inches; and then half its perimeter, or its length and breadth added together muſt be 222 inches.

	1ſt. poſition.	2d poſition.
Breadth	0	1
Length	222	221
	—— ×	—— ×
Area	0	221
	12096	12096
	— ——	— ——
Error	12096	11875

3d poſition.	True poſition.
2	96
220	126
—— ×	—— ×
440	12096
12096	
— ——	
11656	

The reſolving ſeries of the 2d order	12096, 11875, 11656, 11439, &c.
The 1ſt differences	221, 219, 217, &c.
The 2d and conſtant differences	2, 2, &c.
Its component ſeries of the 1ſt order	96, 95, 94, 93, &c. 126, 125, 124, 123, &c.

The roots required $\frac{96}{96-95}=96$ for the breadth,

and $\frac{126}{126-125}=126$ for the length.

Note. This queſtion ſhews how groſly they are miſtaken who think to eſtimate the areas or magnitudes of plain

plain figures by their perimiters, as if such figures were greater or less in proportion as their perimeters are so; whereas here we see that the perimeter of one figure may be greater than that of another by 4 inches, and that at the same time its area may be less than the area of that other by 4 square inches.

Quest. III. There are two numbers whose sum is 22, and the sum of their cubes 2728.

Ans.	1st position.	2d position.
1st number	0	1
2d	22	21
their cubes	0	1
	10648	9261
	10648 +	9262 +
	2728	2728
Error	7920 −	6534 −

	3d position.	True position.
	2	10
	20	12
	8	1000
	8000	1728
	8008 +	2728 +
	2728	
	5280 −	

Resolving series of the 2d order } 7920, 6534, 5280, 4158, 3168, &c.
The 1st diff. 1386, 1254, 1122, 990, &c.
The 2d and constant diff. 132, 132, 132, &c.

Its component series $\begin{cases} \text{of the 1st order } 10, 9, 8, 7, 6, \&c. \\ \text{of the 1st order } 12, 11, 10, 9, 8, \&c. \\ \text{constant } 66, 66, 66, 66, 66, \&c. \end{cases}$

Its roots $\frac{10}{10-9} = 10$ for the 1st number,

And $\frac{12}{12-11} = 12$ for the 2d.

Note. Although this question appears at first sight to be of the 3d degree, it belongs however to the 2d, and therefore it is compounded of three series, two of the 1st order and one constant.

Quest. IV. To find a number, whose square multiplied by the excess of 20 upon the same number sought, gives the product 768.

Ans.	1st position.	2d position.
The number	0	1
	20	20
	20	19
Its square	0	1
	0 ×	19 ×
	768	768
Error	768	749

3d position.	4th position.	True position.
2	3	8
20	20	20
18	17	12
4	9	64
72 ×	153 ×	768 ×
768	768	
696	615	

Resolving series of the 3d order	768,	749,	696,	615,	512,	&c.
The 1st differences		19,	53,	81,	103,	&c.
The 2d differences			−34,	−28,	−22,	&c.
The 3d and const. diff.				−6,	−6,	&c.
Its component series of the 1st order	8,	7,	6,	5,	4,	&c.
Its component series of the 2d order	96,	107,	116,	123,	128,	&c.

Its rational root $\frac{8}{8-7}=8$ for the number required.

Notes. I. In taking the differences of the terms of a series the 2d term must be subtracted from the 1st, the 3d from the 2d, the 4th from the 3d, and so on.

II. The resolving series may be continued as long as you please, by the continual subtraction of the differences.

III. The resolving series are of the same use in Arithmetic as the equations are in Algebra, in order to discover the nature of the question, as well as of the numbers sought, as will appear by the following Examples, in which the centre is distinguished by a little star.

Example I.

Series of the 1st order. (Queſt. I.)

$$\begin{array}{cccccccccccccc} & * & & & & & & & & & & & \\ 44, & 43, & 42, & \ldots\ldots, & 3, & 2, & 1, & 0, & -1, & -2, & -3, & \&c. \\ & 1, & 1, & \ldots\ldots, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & \&c. \end{array}$$

This ſeries is compoſed of two ranks of numbers, both departing from 0; but one towards the right comprehends the numbers −1, −2, −3, &c. affected with the ſign −, and therefore called *negative*; and the other toward the left contains the numbers 1, 2, 3, &c. which for the contrary reaſon, are *poſitive*. The errors being then but once =0, the ſeries has one root only, and conſequently the queſtion has only one anſwer. The root muſt be poſitive, becauſe the centre and its exceſs above the next following term are underſtood to have the ſame ſign, viz. +

Example II.

Series of the 2d order. (Queſt. II.)

$$\begin{array}{ccccccccc} * & & & & & & & & \\ 12096, & 11875,., & 0, & -29,., & -56, & -29, & 0, & 31, & \&c. \\ & 221,., & 31, & 29,., & -25, & -27, & -29, & -31, & \&c. \\ & & 2, 2, & 2,., & 2, & 2, & 2, & 2, & \&c. \end{array}$$

This ſeries is compoſed of three ranks of numbers, one negative in the middle between two cyphers, and the other two at the extremities, both poſitive, equal, and increaſing without end. The errors being then twice =0, the ſeries has two rational and integral roots, and conſequently the queſtion has two anſwers. The roots are both poſitive, becauſe the centre, its exceſs above the next following term, and the conſtant difference, that is, 12096, 221, and 2, are all poſitive. The 1ſt anſwer gives the breadth of the oblong ſquare =96 and length=126; and the 2d anſwer, the breadth =126 and the length=96.

Ex-

Example III.

Series of the 2d order. (Queſt. III.)

7920, 6534, . . . , 198, 0, −66, 0, 198, &c.
1386, . . . , 330, 198, 66, −66, −198, &c.
132, 132, 132, 132, 132, &c.

This ſeries being like the next preceding, needs no explication.

Example IV.

Series of the 3d order. (Queſt. IV.)

−168, ., 768, 749, 696, ., 131, 0, −123, ., 120, &c.
−387, ... 19, 53, ., 133, 131, 123, ., −219, &c.
−82, ... −34, ., −4, 2, 8, ., 62, &c.
−6, ... −6, −6, −6, ., −6, &c.

This ſeries being compoſed of four ranks of numbers, two poſitive and two negative, muſt have three roots; and, becauſe the errors ſuffer three mutations, viz. being at firſt negative, they become poſitive, then again negative, and then again poſitive, the three roots are all poſſible, but only one of them is rational and integral, the errors being only once reduced $=0$. Of theſe roots only one is negative, becauſe the centre, the 1ſt, 2d, and 3d differences ſuffer only a mutation, paſſing once from the poſitive ſtate to the negative, as appears by diſpoſing thoſe quantities, as follows, 768, 19, −34, −6. We will teach in the next following Chapter how to find the two other roots, which are irrational.

CHAP.

CHAPTER XX.

THE RULE OF ROOTS,

WHEREIN OF THE IRRATIONAL ROOTS OF THE SECOND DEGREE.

THE Rule of Roots is a method of finding the roots either rational or irrational, of a series of the second degree.

Rule of Roots. *Take half the constant difference of the series, add it to the excess of the centre above the next following term, and divide the sum by the same half; the result will be the sum of both roots of the series. Then from the square of half this sum subtract the quotient arising from dividing the centre by half the constant difference and from the remainder extract the square root, which added to, and subtracted from, the said half sum, will give the two roots required.*

Note. This Rule is of the same use in Arithmetic, as the resolution of quadratic equations in Algebra.

Quest. 1. What number is that, which being added to its square will make 210?

Ans.

Anf.	1ft pofition.	2d pofition.
The number	0	1
Its fquare	0	1
	+ 0	+ 2
	210	210
Error	210	208

3d pofition.	True pofition.
2	14
4	196
+ 6	+ 210
210	
204	

Refolving feries of the 2d order	210, 208, 204, 198, 190, &c.
The 1ft differences	2, 4, 6, 8, &c.
The 2d and conftant diff.	−2, −2, −2, &c.

Half the conftant difference $= -1$,

The fum of the roots $= \dfrac{210 - 208 - 1}{-1} = \dfrac{1}{-1} = -1$.

The roots required $= -\tfrac{1}{2} \pm \sqrt{\left(\dfrac{1}{4} - \dfrac{210}{-1}\right)}$

$= -\dfrac{1}{2} \pm \sqrt{\left(\dfrac{841}{4}\right)} = -\tfrac{1}{2} \pm \tfrac{29}{2}$, that is, 14 and -15.

Note. The negative root -15 refolves the queftion, if the number ought to be fubtracted from its fquare, becaufe $225 - 15 = 210$.

Queſt. II. I demand a number, which being ſubtracted 12 times from its ſquare leaves 96.

Anſ.	1ſt poſition.	2d poſition.
The number	0	1
	$\times 12$	$\times 12$
	0	12
Its ſquare	0	1
	0	-11
	96	96
Error	96	107

3d poſition.	True poſition.
2	$6\pm\sqrt{132}$
$\times 12$	$\times 12$
24	$72\pm 12\sqrt{132}$
4	$168\pm 12\sqrt{132}$
-20	96
96	
116	

	*					
The reſolving ſeries of the 2d order	96,	107,	116,	123,	128,	&c.
The 1ſt differences		-11,	-9,	-7,	-5,	&c.
The 2d and conſtant diff.			-2,	-2,	-2,	&c.

Half the conſtant difference $= -1$,

the ſum of the roots $= \frac{96-107-1}{-1} = 12$.

The roots required $= 6 \pm \sqrt{\left(36 \frac{-96}{-1}\right)} = 6 \pm \sqrt{132}$, that is, $6+\sqrt{132}$, and $6-\sqrt{132}$.

Note.

Note. This series is the same as the component of the 2d order in Question IV. and therefore the roots $6 \pm \sqrt{132}$ will answer to the said question. Hence it appears, that if a resolving series of the higher order be compounded of some of the 2d, its quadratic roots may be found by the Rule of Roots, after having resolved the said series in its components of the 2d order.

CHAPTER XXI.

THE RULE OF LIMITS,

WHEREIN OF THE ROOTS BY APPROXIMATION.

LIMITS of a Root are called two numbers, between which the root is contained, and the less number is the *first* limit.

The Rule of Limits is a method of finding the irrational roots of a series by approximation, that is, by giving an approximate value of them, but to any degree of exactness required.

Rule of Limits. *Find the integral limits of the root required, either by way of the False Positions, or by the Rule of Series; then add to the first limit any decimals, and find again the decimal limits of the same root, and so on, adding any lower decimals, find the new limits, but to any degree of exactness required.*

Note. By this Rule any root whatsoever of a number may be easily found in decimals, as will appear from the following Question III.

Quest. I. What two numbers are those, whose difference is 4, and the product 8?

Anf.	1st position.	2d position.
The less numb.	0	1
The greater	4	5
	— ×	— ×
Their product	0	5
	8	8
	— —	— —
Error	8	3

3d position.	Position by Approximation.
2	1.46
6	5.46
— ×	——
12	7.9716
8	8.0000
— —	— ——
−4	0.0284

The resolving series of the 2d order	*8,	3,	−4,	−13,	−24,	−37, &c.
The 1st differences		5,	7,	9,	11,	13, &c.
The 2d and constant diff.		−2,	−2,	−2,	−2, &c.	

It is obvious, that the root is between 1 and 2, because the error answering to the position 1 is positive, and the error answering to the position 2 is negative; Therefore taking the positions with decimals, viz. 1.1, 1.2, &c. the errors will be as follows:

Positions	1.1 ,	1.2 ,	1.3 ,	1.4 ,	1.5 ,	&c.	True root between 1.4 & 1.5.
Errors	2.39,	1.76,	1.11,	0.44,	−0.25,	&c.	

Now

Now passing to the positions 1.41, 1.42, &c. the new errors will be as follows:

Positions	1.41	, 1.42	, 1.43	, 1.44	, 1.45	, &c.
Errors	0.3719,	0.3036,	0.2351,	0.1664,	0.0975,	&c.

And so adding to these positions the thousandths, ten-thousandths, &c. the value of the root sought may be determined to any degree of exactness required.

Quest. II. To find two numbers, whose sum is 10 and product 26.

Ans.	1st position.	2d position.
One number	0	1
The other	10	9
	—×	—×
Their product	0	9
	26	26
	——	——
Error	26	17

3d position.	Nearest position.
2	5
8	5
—×	—×
16	25
26	26
——	——
10	1

Resolving series of the 2d degree	26,	17,	10,	5,	2,	1,	2,	5, &c.
The 1st differences	9,	7,	5,	3,	1,	−1,	−3,	&c.
The 2d and constant diff.	2,	2,	2,	2,	2,	2,	&c.	

This series being composed of two ranks of numbers, both positive, and increasing without end, teaches us that the errors cannot be destroyed, and therefore

 that

that series has no real roots and the question no answer. The smallest error which can be made is $=1$, and then the position is $=5$, which cannot be reduced to greater exactness.

Quest. III. To extract any root whatsoever from a given number.

Ans. If the root is rational, it will be found either by the Rule of Divisors, or by the Rule of Series. But, if it is irrational, the Rule of Limits will give an approximate value of it.

Example I. What is the cube root of 3375?

Suppose 12; the cube of 12 is 1728, and therefore the error $3375-1728=1647$, whose divisors being 1, 3, 9, 27, 61, &c. the corrections are 13, 15, 21, 39, 73, &c. Now it is evident that 15 only may be the rational root, because only its cube can end in 5; and really 15 is the root required.

Example II. What is the cube root of 4296?

Its first limit is 16, and its root by approximation in hundredths is 16.25, as it appears from the following calculation:

Positions	16.1	, 16.2	, 16.3	, &c.
Errors	122.719,	44.472,	-34.747,	&c.
Positions	16.21	, . . , 16.25	, 16.26	, &c.
Errors	36.593939, . . ,	4.984375,	-2.942376,	&c.

CHAP.

CHAPTER XXII.

A PROMISCUOUS COLLECTION OF QUESTIONS.

1. A was born when B was 21 years of age; how old will A be when B is 47, and what will be the age of B when A is 60? Anſ. A 26, B 81.

2. A perſon at the time of his out-ſetting in trade, owed 350*l.* and had in caſh 5307*l.* 10*s.* in wares 713*l.* 7*d.* and in good debts 210*l.* 5*s.* 10*d.* Now after having traded a year he owed 703*l.* 17*s.* and had in caſh 4874*l.* 9*s.* 4*d.* in bills 350*l.* in wares 1075*l.* 14*s.* 3½*d.* and in recoverable debts 613*l.* 13*s.* 10½*d.* What was his real gain that year? Anſ. 329*l.* 4*s.* 1*d.*

3. A gentleman's daily expence is 4*l.* 8*s.* $1\frac{19}{365}$*d.* and he ſaves 500*l.* in the year; what is his yearly income? Anſ. 2107*l.* 12*s.*

4. Having a piece of land 11 poles in breadth, I demand what length of it muſt be taken to contain an acre, when four poles in breadth require 40 poles in length to contain the ſame? Anſ. 14 poles, 3 yds.

5. If a gentleman, whoſe annual income is 1000*l.*, ſpend 20 guineas a week, whether will he ſave or run in debt, and how much in the year? Anſ. 92*l.* debt.

6. In the latitude of London, the diſtance round the earth, meaſuring in the parallel of latitude is about 15550 miles, now as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried by this motion from weſt to eaſt?

Anſ. $649\frac{259}{359}$ miles an hour.

7. In

7. In order to raise a joint stock of 10000*l.* A, B, and C, together subscribe 7950*l.* and D the rest; now A and B are known to have set their hands to 5800*l.* and A has been heard to say that he had undertaken for 550*l.* more than B; what did each proprietor advance? Ans. A 3175, B 2625, C 2150, D 2050.

8. Divide 1000 crowns, give A 120 more, and B 95 less than C. Ans. A 445, B 230, C 325.

9. Laid out 165*l.* 15*s.* in wine at 4*s.* 3*d.* a gallon, some of which receiving damage in carriage, I sold the rest at 6*s.* 4*d.* a gallon, which produced only 110*l.* 16*s.* 8*d.* what quantity was damaged? Ans. 433 gal.

10. A father divided his fortune among his sons, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share were 5000*l.* Ans. 11875*l.*

11. A stationer sold quills at 10*s.* 6*d.* a thousand, by which he cleared $\frac{1}{3}$ of the money; but growing scarce, raised them to 12*s.* a thousand; what did he clear per cent. by the latter price? Ans. 71*l.* 8*s.* $6\frac{6}{7}$*d.*

12. If 1000 men besieged in a town, with provisions for 5 weeks, allowing each man 16 ounces a day, were reinforced with 500 men more; and hearing that they cannot be relieved till the end of 8 weeks; how many ounces a day must each man have, that the provision may last that time? Ans. $6\frac{2}{3}$ oz.

13. If a quantity of provisions serve 1500 men 12 weeks, at the rate of 20 ounces a day for each man; how many men will the same provisions maintain for 20 weeks at the rate of 8 oz. a day for each man?

Ans. 2250 men.

14. In what time will the interest of 72*l.* 12*s.* equal that of 15*l.* 5*s.* for 64 days, at any rate of interest? Ans. $13\frac{161}{363}$ days.

15. A person possessed of $\frac{1}{4}$ of a ship, sold $\frac{2}{3}$ of his share for 1260*l.* what was the reputed value of the whole at the same rate? Ans. 5040*l.*

16. A

16. A father dying left his ſon a fortune, $\frac{1}{4}$ of which he ran through in 8 months, $\frac{3}{7}$ of the remainder laſted him a twelvemonth longer, after which he had bare 410*l.* left; what did his father bequeath him?

Anſ. 956*l.* 13*s.* 4*d.*

17. A perſon being aſked the hour of the day, ſaid, The time paſt noon is equal to $\frac{4}{5}$ths of the time till midnight; what was the time? Anſ. 20 min. paſt 5.

18. A perſon looking on his watch, was aſked what was the time of the day, who anſwered, It is between 4 and 5; but a more particular anſwer being required, he ſaid that the hour and minute hands were then exactly together; what was the time?

Anſ. 21$\frac{9}{11}$ min. paſt 4.

19. With 12 gal. of canary at 6*s.* 4*d.* a gal. I mixed 18 gal. of white wine at 4*s.* 10*d.* a gal. and 12 gal. of cyder at 3*s.* 1*d.* a gal. at what rate muſt I ſell a quart of this compoſition ſo as to clear 10 per cent.

Anſ. 1*s.* 3$\frac{5}{7}$*d.*

20. Suppoſe that I have $\frac{3}{16}$ of a ſhip worth 1200*l.* what part of her have I left after ſelling $\frac{2}{5}$ of $\frac{4}{9}$ of my ſhare, and what is it worth? Anſ. $\frac{37}{240}$, worth 185*l.*

21. What length muſt be cut off a board 8$\frac{3}{8}$ inches broad to contain a ſquare foot, or as much as 12 inches in length and 12 in breadth? Anſ. 17$\frac{13}{67}$ inches.

22. If, by ſelling goods at 50*s.* per cwt. I gain 20 per cent. what do I gain or loſe per cent. by ſelling at 45*s.* per cwt.? Anſ. 8*l.* gain.

23. Sold goods for 60 guineas, and by ſo doing loſt 17 per cent. whereas I ought in dealing, to have cleared 20 per cent. then how much under their juſt value were they ſold? Anſ. 28*l.* 1*s.* 8$\frac{20}{83}$*d.*

24. If, by ſelling goods at 27*d.* per lb. I gain cent. per cent. what do I clear per cent. by ſelling for 9 guineas per cwt. Anſ. 50 per cent.

25. If

25. If 20 men can perform a piece of work in 12 days, how many will accomplish another thrice as big in one fifth of the time? Anf. 300.

26. A perfon making his will gave to one child $\frac{13}{20}$ of his eftate, ane the reft to another, add when thefe legacies came to be paid, the one turned out 600*l.* more than the other; what did the teftator die worth?

Anf. 2000*l.*

27. A father devifed $\frac{7}{18}$ of his eftate to one of his fons, and $\frac{7}{18}$ of the refidue to another, and the furplus to his relict for life; the children's legacies were found to be 257*l.* 3*s.* 4*d.* different; pray what money did he leave the widow the ufe of? Anf. 635*l.* $10\frac{30}{49}$*d.*

28. There is a number which, if multiplied by $\frac{2}{3}$ of $\frac{3}{4}$ of $1\frac{3}{4}$, will produce 1; what is the fquare of that number? Anf. $1\frac{15}{49}$.

29. A perfon dying left his wife with child, and making his will, ordered that if fhe went with a fon, $\frac{2}{3}$ of his eftate fhould belong to him, and the remainder to his mother; and if fhe went with a daughter, he appointed the mother $\frac{2}{3}$ and the girl the remainder; but it happened that fhe was delivered both of a fon and a daughter, by which fhe loft in equity 240*l.* more than if it had been only a girl; what would have been her dowry had fhe had only a fon?

Anf. 2100*l.*

30. Three perfons purchafe together a fhip, toward the payment of which A advanced $\frac{2}{9}$, and B $\frac{2}{7}$ of the value, and C 200*l.* how much paid A and B, and what part of the veffel had C?

Anf. A $90\frac{10}{31}$*l.* B $116\frac{4}{31}$*l.* C $\frac{31}{63}$ part.

31. A and B clear by an adventure at fea 63 guineas, with which they agree to buy a horfe and chaife, of which they were to have the ufe in proportion to the fums adventured, which was found to be A 9 to B 8,

B 8, they cleared 45 per cent. what money then did each send abroad?

Anſ. A 74*l.* 2*s.* 4$\frac{4}{17}$*d.* and B 65*l.* 17*s.* 7$\frac{13}{17}$*d.*

32. Three perſons entered joint trade, to which A contributed 240*l.* and B 210*l.* they clear 120*l.* of which 30*l.* belongs of right to C; required that perſon's ſtock, and the ſeveral gains of the other two?

Anſ. C's ſtock 150*l.* A gained 48*l.* and B 42*l.*

33. A clears 12*l.* in 6 months, B 15*l.* in 5 months, and C, whoſe ſtock was 40*l.* clears 21*l.* in 9 months; what was the whole ſtock? Anſ. 125$\frac{5}{7}$*l.*

34. A had 12 pipes of wine which he parted with to B at 4¼ per cent. profit, who ſold them to C for 40*l.* 12*s.* advantage; C made them over to D for 605*l.* 10*s.* and cleared thereby 6 per cent. how much a gallon did the wine coſt A? Anſ. 6*s.* 8$\frac{6640}{11077}$*d.*

35. A of Amſterdam orders B of London to remit to C of Paris, at 52½*d.* ſterling a crown, and to draw on D of Antwerp for the value at 34½*s.* Flem. a pound ſter. but as ſoon as B received the commiſſion, the exchange was on Paris at 53*d.* a crown; pray at what rate of exchange ought B to draw on D to execute his orders and be no loſer? Anſ. 34*s.* 2$\frac{5}{33}$*d.*

36. A, with intention to clear 20 guineas on a bargain with B, rates hops at 15*d.* a lb. which coſt him 10½*d.* B, appriſed of that, ſets down malt, which coſt 20*s.* a quarter, at an adequate price; for how much malt did they contract? Anſ. 49 qr.

37. A and B venturing equal ſums of money, clear by joint trade 180*l.* By agreement A was to have 8 per cent. becauſe he ſpent time in the execution of the project, and B was to have only 5; what was allotted to A for his trouble? Anſ. 41*l.* 10*s.* 9$\frac{3}{13}$*d.*

38. Laid out in a lot of muſlin 500*l.* upon examination of which 3 parts in 9 proved damaged, ſo that I could make but 5*s.* a yard of the ſame; and by ſo

 doing

doing find I loſt 50*l.* by it; at what rate per ell am I to part with the undamaged muſlin, in order to gain 50*l.* upon the whole? Anſ. 11*s.* $7\frac{2}{7}$*d.*

39. A at Paris draws on B in London 1400 crowns, at 46*d.* ſterling a crown, for the value of which B draws again on A at 57*d.* ſterling a crown, beſides reckoning commiſſion $\frac{1}{2}$ per cent. Did A gain or loſe by this tranſaction, and what?

Anſ. He gained $17\frac{13}{19}$ crowns.

40. A, B, and C are in company, A put in his ſhare of the ſtock for 6 months, and laid claim to $\frac{1}{6}$ of the profits, B put in his for 9 months, C advanced 500*l.* for 8 months, and required on the balance $\frac{3}{5}$ of the gain; required the ſtock of the two other adventurers?

Anſ. A 185*l.* 3*s.* $8\frac{4}{9}$*d.* and B 172*l.* 16*s.* $9\frac{13}{27}$*d.*

41. A young hare ſtarts 40 yards before a greyhound and is not perceived by him till ſhe has been up 40 ſeconds, ſhe ſcuds away at the rate of 10 miles an hour, and the dog, on view, makes after her at the rate of 18; how long will the courſe hold, and what ground will be run over, beginning with the out-ſetting of the dog? Anſ. $60\frac{5}{27}$ ſec. and 530 yards run.

42. If I leave Exeter at 8 o'clock on Monday morning for London, and ride at the rate of 3 miles an hour without intermiſſion, and B ſet out from London for Exeter at 4 the ſame evening, and ride 4 miles an hour conſtantly; ſuppoſing the diſtance between the two cities to be 130 miles, whereabout on the road ſhall they meet? Anſ. $69\frac{3}{7}$ miles from Exeter.

43. A reſervoir for water has two cocks to ſupply it; by the firſt alone it may be filled in 40 minutes, by the ſecond in 50 minutes, and it hath a diſcharging cock, by which it may, when full, be emptied in 25 min. Now, ſuppoſing that theſe cocks are all left open, and that the water comes in, in what time, ſuppoſing the influx and efflux to be always alike, would the ciſtern be filled? Anſ. 3 hrs. 20 min.

44. A

44. A ſets out from London for Lincoln at the very ſame time that B at Lincoln ſets forward for London, diſtant 100 miles, after 7 hours they meet on the road, and it then appeared that A had rode $1\frac{1}{2}$ miles an hour more than B; at what rate an hour did each of them travel? Anſ. A $7\frac{25}{28}$, and B $6\frac{11}{28}$ miles.

45. A and B truck, A has $12\frac{1}{2}$ cwt. of Farnham hops at 2*l.* 16*s.* a cwt. but in barter inſiſts on 3*l.* B has wine worth 5*s.* a gallon, which he raiſes in proportion to A's demand; on the balance A received but a hhd. of wine; what had he in ready money?

Anſ. 20*l.* 12*s.* 6*d.*

46. A of Amſterdam owes to B of Paris 3000 guilders of current ſpecie, which he is to remit to him by order, the exchange 91*d.* Flemiſh de banco a crown, the agio 4 per cent. but when this was to be negociated, the exchange was down at 90*d.* a crown, and the agio 5 per cent, what did B get by this turn of affairs? Anſ. 5 liv. 12 ſol. $8\frac{584}{1183}$ den.

47. If 100 eggs be laid down on the ground in a ſtraight line, one yard from each other, and the firſt of them one yard from a baſket; what ſpace ſhall a man walk over in bringing the eggs one by one to the baſket?

Anſ. 10100 yds. or 5 miles, 1300 yds.

PART II.

THE ARITHMETICO - ALGEBRAICAL LANGUAGE.

CHAPTER I.

THE NATURE OF THIS LANGUAGE, AND DIVISION OF QUESTIONS BELONGING TO IT.

I. *The Nature of the Arithmetico-Algebraical Language.*

THE arithmetico-algebraical language is an application of the *analytical art* to the resolution of those questions where the quantities are expressed by numbers, and their magnitude alone is to be considered.

The

The *analytical art*, *resolution*, or *analysis*, generally speaking, is the method of proceeding from the thing *sought* or *taken* for granted through its consequences to something that is *really* granted: but here it is strictly considered as a method of investigating quantities that are unknown, from *certain given relations* to each other, and to such as are known; which relations being almost universally either that of equality, or such as may be reduced to that of equality, are expressed by equations representing more conveniently and more distinctly the conditions of the question when translated out of common language into the arithmetico-algebraical; and because these equations must contain one or more unknown quantities, the principal business of this language will be the deducing *final equations* containing only one unknown quantity and resolving them.

Hence the nature of this language is all comprehended in the following

General Principle.

The *unknown* quantities in the question proposed must be expressed by *letters*, and the *relations* of the *known* and *unknown* quantities contained in it, or the *conditions* of it as they are called, must be expressed by *equations*. These equations being resolved by their proper rules, will give the *answer* of the question.

II. *Division of Questions.*

The division of questions arises from four different principles; 1st, from their nature; 2dly, from the number of the unknown quantities in relation to the number of the conditions; 3dly, from the highest power of the unknown quantity to be found in any term

term of final equation; and 4thly, from the number of the letters employed to express the unknown quantities which are to be found.

First Division.

Questions by their nature are either *possible* or *impossible*. A question is *possible* when its conditions are not found to be inconsistent, either with one another or with the principles of our reasoning, or with the nature of things which are treated of in the question; and on the the contrary, a question is *impossible* when there are no conditions affording any equation, or when the conditions are inconsistent, either with one another, with the principles of our reasoning, or with the nature of the things which are treated of. I will explain this division by the following Examples:

Example I. What are two numbers, whose sum is 20 and difference 10?

These conditions are not inconsistent with one another, and no absurdity is to be found; therefore the numbers sought are 15 and 5.

Example II. To find two numbers, one of which is double of the other, and the third part of the greater double of the fifth part of the less number.

These conditions are inconsistent with one another, because if a number is double of another, the third part of the greater must be double of the third, and not of the fifth part of the less number.

Example

Example III. I demand a number whose half may be double of its third.

This condition is inconsistent with the principles of our reasoning, by which we are instructed, that a half of any number is double of its quarter, and therefore it cannot be double of its third.

Example IV. To buy 20 horses, some black and some white, so that the white may be 3 more than the black, how many must I purchase of each sort?

The answer being 11½ white and 8½ black, shews evidently that the conditions are inconsistent with the nature of horses, which are indivisible.

Example V. What is the number I carry now in my mind?

This question affording no condition proper to its resolution, becomes impossible.

Second Division.

Considering the number of the unknown quantities in relation to the number of the conditions, three cases are to be found; for the question may comprehend *expressly* or *implicitly*, 1st, as many *independent* conditions as there are unknown quantities to be discovered by them; 2dly, more independent conditions than unknown quantities; and 3dly, more unknown quantities than independent conditions.

I said, that so many independent conditions ought to be comprehended in the question *expressly* or *implicitly*, because it may happen that a condition may not be expressed in a question and yet be implied in the nature of

of the thing; thus, in a queſtion, where ſeveral rods are to be ſet upright in a ſtraight line at certain intervals, it is implied, though not expreſſed, that the number of intervals muſt be leſs than the number of rods by unity. See Chap. V. Sect. I. Queſt. X.

Sometimes a condition may be introduced into a queſtion that includes two or more conditions; as, when we ſay, four numbers are in continual proportion, we mean, not only that the firſt number is to the ſecond, as the ſecond is to the third, but alſo, that the ſecond is to the third as the third is to the fourth.

Laſtly, it is to be obſerved, that the conditions, and hence the equations expreſſing them, muſt be independent, that is, the one muſt not be deducible from the other by any mathematical reaſoning; for, otherwiſe, there would in effect be only one equation under two different forms, from which ſo ſolution can be derived.

Caſe I. When there are as many independent conditions as there are unknown quantities, the queſtion is juſtly propoſed, and called *determinate*, becauſe by its nature it is determined to admit of a finite number of anſwers, although by its conditions the anſwers may be found ſometimes impoſſible. Examples will be given at their proper places. (See the queſtions of Chap. VI.)

Caſe II. When there are more independent conditions than unknown quantities, any condition will be either unneceſſary or inconſiſtent with the others; and therefore the queſtion, which is *more than determinate*, will contain either a theorem, or an impoſſible problem. Suppoſe it is required to find a number, half of which is 5, and its fifth $=2$; the number is 10, and the queſtion contains a theorem, becauſe, if $\frac{10}{2}=5$, it ought to be $\frac{10}{5}=2$. But, if half of the number ſought muſt be 5, and its fifth $=3$, the queſtion becomes impoſſi-

 ble,

ble, becauſe the number anſwering to the firſt condition is but 10, and the number required by the ſecond condition is 15.

Caſe III. Laſtly, when there are more unknown quantities than independent conditions, then the queſtion is called *indeterminate*, becauſe it may admit of an infinite number of anſwers, ſince the conditions wanting may be aſſumed at pleaſure. There may be other circumſtances, however, to limit the anſwers to one or a preciſe number, and which at the ſame time cannot be directly expreſſed by equations. Such are theſe, that the numbers muſt be integers, ſquares, cubes, and many others. The ſolution of ſuch queſtions which are alſo called *diophantine*, ſhall be conſidered afterwards. (See Chap. VII.)

Third Diviſion.

If we conſider final equations, or equations containing one unknown quantity and its powers; theſe equations, as well as the queſtions producing them are divided into *orders* or *degrees*, according to the higheſt power of the unknown quantity to be found in any of the terms of the ſaid equations.

If the higheſt power of the unknown quantity in any term be the 1ſt, 2d, 3d, &c. the equation is called *ſimple*, *quadratic*, *cubic*, &c. and the queſtion of the 1ſt, 2d, 3d, &c. *order* or *degree*.

But the exponents of the unknown quantity are ſuppoſed to be integers, and the equation is ſuppoſed to be cleared of fractions, in which the unknown quantity, or any of its powers, enter the denominators. Thus $x+5=\frac{6x-15}{3}$ is a ſimple equation; $3x-\frac{5}{2x}=12$, when cleared

cleared of the fraction by multiplying both fides by $2x$, becomes $6x^2-5=24x$ a quadratic; $x^3-2x^4=x^6-20$ is an equation of the 6th order or degree, &c.

Fourth Divifion.

Laftly, queftions are faid to be with *one* unknown quantity, with *two*, with *three*, with *four*, &c. unknown quantities, as there are one, two, three, four, &c. letters employed to exprefs them.

Note. It is often eafy to employ fewer letters than there are unkuown quantities, by expreffing fome of them from a fimple relation to others contained in the conditions of the queftions, which confequently muft receive their name, not from the number of the unknown quantities, but from the number of letters employed to exprefs them. (See Queftions II, III, VIII, and XX in the Note.)

CHAPTER II.

OF WRITING ARITHMETICO-ALGEBRAICALLY

WE are faid to *write arithmetico-algebraically* when by figures, letters, and mathematical figns we reprefent the known and unknown quantities in the queftion propofed, as well as their fum, difference, product, quotient, powers, roots, and compound operations, re-

quired in it; and also, when by equations we express the conditions of the question, that is, the relations of the known and unknown quantities, contained in it.

The principal business of this Chapter will therefore be the fundamental operations upon the known and unknown quantities, and the expression of questions, that is the translating of them from common language into that of analysis.

I. *Fundamental Operations.*

Definitions. I. The *juxta position* of letters as in the same word, expresses the product of the quantities denoted by these letters. Thus xy expresses the product of x and y; xyz expresses the continued product of x, y, and z.

II. A number prefixed to a letter is called a *numeral coefficient*, and expresses the product of the quantity by that number, or how often the quantity denoted by the letter is to be taken. Thus 3 is the coefficient of x in the product $3x$, and it expresses that the unknown quantity x must be taken three times.

III. The *quotient* of two quantities is denoted by placing the *dividend* above a small line, and the *divisor* below it, as in numbers. Thus $\frac{x}{y}$ is the quotient of x divided by y. This expression of a quotient is also called a *fraction*.

IV. A quantity is said to be *simple* which consists of one part or *term* as x or $-x$, xy, $3xy$; and a quantity is said to be *compound* when it consists of more than one term, connected by the signs $+$ and $-$, as in numbers. If there are two terms, as $x+2y$, it is called a *binomial*; if three, as $x+2y-3$, a *trinomial*, &c.

V. Sim-

V. Simple quantities, or the terms of compound quantities, are ſaid to be *like*, which conſiſt of the ſame letter or letters equally repeated. Thus x and $3x$, xy and $-3xy$, xx and $2xx$, xxy and $-4xxy$, are like quantities; but x and y, $3x$ and xy, $2xx$ and $3xxx$, $2xxy$ and $3xyy$, are unlike.

Notes. I. When a product is expreſſed by the ſame letter repeated it is called a power, and it is expreſſed by an exponent, as in numbers. Thus $xx = x^2$, $xxx = x^3$, &c.

II. When a term has no numeral coefficient unity is underſtood. Thus $x = 1x$, $xy = 1xy$, &c.

III. Becauſe the ſign $\div$ is alſo a mark of diviſion, it muſt be $\frac{x}{y} = x \div y$.

IV. Since $x^3 = xxx = x \times xx = x^1 \times x^2$, powers of the ſame root are multiplied by adding their exponents.

V. And becauſe $x^1 = \frac{x^3}{x^2}$, powers of the ſame root are divided by ſubtracting their exponents.

VI. Laſtly, being $x \times yz = xyz$, and conſequently $x = \frac{xyz}{yz}$, in the diviſion, the letters common both to the dividend and diviſor are to be expunged out of them. Thus $\frac{xyy}{yz} = \frac{xy}{z}$.

General Rule.

The fundamental operations upon the quantities follow the ſame Rules as the compound operations upon the numbers (See Part I. Chap. V. Sect. VI.) *but ſubjoining to the reſult of the ſigns and coefficients the reſult of the letters,*

fars, according to the foregoing Definitions and Notes, as may be ſeen in the following

Examples,

Addition.

$$
\begin{array}{l}
x^2+xy+5x+5y-6z-10 \\
x^2-xy+3x-2y+4z-15 \\
\hline
2x^2 \quad * +8x+3y-2z-25
\end{array}
+
$$

Subtraction.

$$
\begin{array}{l}
x^2+xy+5x+5y-\ \ 6z-10 \\
x^2-xy+3x-2y+\ \ 4z-15 \\
\hline
* \quad 2xy+2x+7y-10z+\ \ 5
\end{array}
-
$$

Multiplication.

First
$$
\begin{array}{r}
x+\ y \\
x+\ y \\
\hline
xx+\ xy \qquad \\
xy+yy \\
\hline
xx+2xy+yy
\end{array}
\begin{array}{l}
\\ \\ \times \\ \\ \\ + \\ \\
\end{array}
$$

Second
$$
\begin{array}{r}
x+y \qquad \\
x-y \qquad \\
\hline
x^2+xy \qquad \\
-xy-y^2 \\
\hline
x^2 \quad * \quad -y^2
\end{array}
\begin{array}{l}
\\ \\ \times \\ \\ \\ + \\ \\
\end{array}
$$

Third
$$
\begin{array}{l}
x^3+2x^2y-3xy^2 \\
\quad x^2-3xy-4y^2 \\
\hline
x^5+2x^4y-3x^3y^2 \\
\quad -3x^4y-6x^3y^2+9x^2y^3 \\
\qquad\qquad -4x^3y^2-8x^2y^3+12xy^4 \\
\hline
x^5-\ \ x^4y-13x^3y^2+x^2y^3+12xy^4
\end{array}
\begin{array}{l}
\\ \\ \times \\ \\ \\ \\ + \\ \\
\end{array}
$$

Di-

Division.

First.

$$
\begin{array}{llll}
x+y\ ) & xx+2xy+yy & (& x+y \\
 & \underline{xx+\ xy\qquad} & & \\
 & *\quad xy+yy & & \\
 & \quad \underline{xy+yy} & & \\
 & \quad *\quad\ * & &
\end{array}
$$

Second.

$$
\begin{array}{llll}
x+y\ ) & x^2 \qquad -y^2 & (& x-y \\
 & \underline{x^2+xy\qquad} & & \\
 & *\ -xy-y^2 & & \\
 & \ \underline{-xy-y^2} & & \\
 & \quad *\quad * & &
\end{array}
$$

Third.

$$
\begin{array}{llllll}
x^3+2x^2y-3xy^2\ ) & x^5 & -\ x^4y & -13x^3y^2 & +x^2y^3 & +12xy^4\ (\ x^2-3xy-4y^2 \\
 & x^5 & +2x^4y & -\ \ 3x^3y^2 & & \\
\hline
 & * & -3x^4y & -10x^3y^2 & +\ \ x^2y^3 & +12xy^4 \\
 & & -3x^4y & -\ \ 6x^3y^2 & +9x^2y^3 & \\
\hline
 & & * & -4x^3y^2 & -8x^2y^3 & +12xy^4 \\
 & & & -4x^3y^2 & -8x^2y^3 & +12xy^4 \\
\hline
 & & & * & * & *
\end{array}
$$

Note. The terms of the dividend are to be ranged according to fome one of its letters, and thofe of the divifor according to the fame letter, as did in the examples according to the powers of *x*.

II. *Ex-*

II. *Expreſſion of Queſtions.*

In this way of notation it is required to ſubſtitute any letters only for ſuch quantities as are unknown, and to expreſs by equations the conditions from which they are to be inveſtigated.

For example, if the queſtion is concerning two numbers, they may be called x and y

If it be required that the ſum of the two numbers ſought be 60, that condition is expreſſed thus	$x+y=60$
If their difference muſt be 24, then	$x-y=24$
If their product is 1640, then	$xy=1640$
If their quotient muſt be 6, then	$x\div y=6$, or $\dfrac{x}{y}=6$
If their ratio is as 3 to 2, then	$x:y::3:2$, and hence $2x=3y$
If the ſum of their ſquares is 100, then	$x^2+y^2=100$
If the difference of their ſquares is 28, then	$x^2-y^2=28$
If the product of their ſquares is 36, then	$x^2y^2=36$
If the quotient of their ſquares is 4, then	$\dfrac{x^2}{y^2}=4$, or $x^2\div y^2=4$
If the product of their ſum multiplied by their difference muſt be 84, then	$\overline{x+y}\times\overline{x-y}=84$, or $x^2-y^2=84$

If the quotient of their ſum divided by their difference ought to be 14, then	$\frac{x+y}{x-y}=14$
If two thirds of one and four ſevenths of the other make 60, then	$\frac{2x}{3}+\frac{4y}{7}=60$
If two thirds of one ſubtracted from four ſevenths of the other leave 11, then	$\frac{4y}{7}-\frac{2x}{3}=11$
If their ſum muſt be 4 times their difference, then	$x+y=4\times\overline{x-y}$, or $x+y=4x-4y$
If the ſum of their ſquares is five times the ſum of the numbers, then	$x^2+y^2=5\times\overline{x+y}$, or $x^2+y^2=5x+5y$
If their product is 6 times their ſum, then	$xy=6\times\overline{x+y}$, or $xy=6x+6y$
If their product is 9 times their quotient, then	$xy=9x\div y$, or $xy=\frac{9x}{y}$
If one number muſt be as much above 20 as the other wants of 20, then	$x-20=\overline{20-y}$
If one number muſt be three times as much above 20 as the other wants of 20, then	$x-20=3\times\overline{20-y}$, or $x-20=60-3y$
If one number increaſed by 12, muſt be to the other, increaſed by 8, as 3 to 4, then	$x+12 : y+8 :: 3 : 4$ hence $4x+48=3y+24$
If their difference, ſum, and product are to each other as are the numbers two, three, and five reſpectively, then	$x-y : x+y :: 2 : 3$ whence $3x-3y=2x+2y$; $x+y : xy :: 3 : 5$ whence $5x+5y=3xy$

If

If one number ought to be as many times contained in 20 as the other contains the number 4, then	$\frac{20}{x} = \frac{y}{4}$ or $20 : x :: y : 4$, whence $xy = 80$
If the number 20 must be a mean proportional between the two numbers sought, then	$x : 20 :: 20 : y$ whence $xy = 400$
If the greater being divided by the less, and again the less by the greater, the first quotient must be to the second as 5 to 3, then	$\frac{x}{y} : \frac{y}{x} :: 5 : 3$ whence $\frac{3x}{y} = \frac{5y}{x}$
If one number increased by 2, and multiplied by the other diminished by 3, produce 40, then	$\overline{x+2} \times \overline{y-3} = 40$ or $xy - 3x + 2y - 6 = 40$

These are some of the relations which are most easily expressed; many others occur, which are less obvious, but as they cannot be described in particular rules, their expression is best explained by examples, and must be acquired by experience.

It has been remarked that it is often easy to employ fewer letters than there are unknown quantities, by expressing some of them from a simple relation to others contained in the conditions of the question. Thus, the solution becomes more easy and elegant. There are some examples of this kind of notation.

Conditions.	Notation.
The sum of the two numbers sought is 60	x and $60 - x$
Their difference is 24.	x and $x + 24$
Their product is 1640.	x and $1640 \div x$

 Their

Their quotient is 6.	x and $x \div 6$
Their ratio is as 3 to 2.	x and $2x \div 3$
The greater is 4 times the lefs.	x and $4x$
One with half the other makes 20.	x and $40 - 2x$
Their fum is twice their difference.	x and $3x$

CHAPTER III.

OF READING ARITHMETICO-ALGEBRAICALLY.

DEFINITIONS. I. When a quantity ftands alone upon one fide of an equation, the quantities on the other fide are faid to be a *value* of it. Thus in the equation $x = 60 - y$, x ftands alone on one fide, and $60 - y$ is a value of it.

II. When an unknown quantity is made to ftand alone on one fide of an equation, and there are only known quantities on the other, that equation is faid to be *refolved*; and the value of the unknown quantity is called a *root* of the equation. Thus the equation $x = 20$ is refolved, and 20 is its root.

If

III. If there are some equations, as A, B, C, D, E, F, containing the same value of the unknown quantity x, it is manifest, that the value is immediately and evidently, without any reasoning, perceived in the equation F, but not so in the others E, D, C, B, A, which gradually require a reasoning more and more complicated, so that it would be very difficult to detect that value in the equation A. The difficulty increases as the number of the unknown quantities and of the equations is greater, as may be seen in the equations G and H, in which x and y have the same value expressed by the equations I and K: but if there may be some methods of drawing from the equation A the equation F, and from the equations G and H the equations I and K, and in general from any number of equations some others of the same form as F, I and K, then the values of the unknown quantities would be manifestly perceived. The use of these methods are what we call here to *read arithmetico-algebraically*, because we are enabled by them to read the value of the unknown quantities, although concealed in very complicated equations.

A	$\frac{2x}{3}+\frac{5}{2}=30-\frac{x}{4}$
B	$8x+30=360-3x$
C	$8x+3x=360-30$
D	$11x=330$
E	$x=\frac{330}{11}$
F	$x=30$
G	$\frac{2x}{7}+\frac{5y}{3}=16$
H	$\frac{5x}{2}-\frac{7y}{6}=28$
I	$x=14$
K	$y=6$

The principal object of this Chapter will therefore be the reducing final equations and resolving them.

SECTION I.

RESOLUTION OF SIMPLE EQUATIONS.

I. *Resolution of Simple Equations, containing only one unknown Quantity.*

SIMPLE equations are resolved by the four fundamental operations already explained, and the application of them to this purpose is contained in the follow-Rules.

RULE I. *To take away the fractions from an equation, multiply the equation, that is, each term of it, by the least multiple of all the denominators, reducing each fraction to its whole.*

Example.

$$\frac{2x}{3}+\frac{5}{2}=30-\frac{x}{4} \quad \times 12$$

$$8x+30=360-3x$$

RULE II. *To bring all the unknown terms on one side of the equation and all the known on the other, transpose all the unknown terms to one side and all the known ones to the other, changing the signs of the terms you transpose.*

Ex-

Example.

If $8x+30=360-3x$, then $8x+3x=360-30$.

RULE III. *To reduce an equation to its simple expression, do the operations that the several signs demand.*

Example.

If $8x+3x=360-30$, then $11x=330$.

RULE IV. *To take away the coefficient or multiplier of the unknown quantity, divide the other side of the equation by it.*

Example.

If $11x=330$, then $x=\frac{330}{11}$, or $x=30$.

General Rule.

Any simple final equation may be resolved by these Rules in the following manner:

First, *Take away the fractions by* Rule I. Secondly, *bring all the unknown terms to one and the same side*, viz. *to that side, which after reduction will exhibit the unknown quantity affirmative, and all the known terms to the other by* Rule II. Thirdly, *reduce the equation to its simple expression by* Rule III. Lastly, *find the root of the equation by* Rule IV.

Example I.

$$5x - \frac{5x}{2} + 12 = \frac{4x}{3} + 26 \quad \times 6$$

(Rule I.) $30x - 15x + 72 = 8x + 156$

(Rule II.) $30x - 15x - 8x = 156 - 72$

(Rule III.) $7x = 84$

(Rule IV.) $x = \frac{84}{7} = 12$

Example II.

$$\frac{7x}{8} - 5 = \frac{9x}{10} - 8 \quad \times 40$$

(Rule I.) $35x - 200 = 36x - 320$

(Rule II.) $320 - 200 = 36x - 35x$

(Rule III.) $120 = x$

II. *Resolution of Simple Equations involving two unknown Quantities.*

These equations may be resolved by any of the following Rules.

RULE I. *A value of one of the unknown quantities must be derived from each of the equations; and these two values being put equal to each other, a new equation will arise, involving only one unknown quantity, and may therefore be resolved by the preceding general Rule.*

III. Ex-

Example.

1st equation $5x-3y=90$; 2d equation $2x+5y=160$

$$5x=90+3y \qquad 2x=160-5y$$

$$x=\frac{90+3y}{5} \qquad x=\frac{160-5y}{2}$$

Hence $\dfrac{90+3y}{5}=\dfrac{160-5y}{2}$

$\times 10$

$180+6y=800-25y$, or $31y=620$, and $y=20$.

Again, because $5x-3y=90$ and $3y=60$,
it will be $5x-60=90$ and $x=30$.

RULE II. *A value of one of the unknown quantities must be derived from one of the equations, and substituted instead of the same unknown quantity in the other equation, which will contain only one unknown quantity, the value whereof may be found as usual.*

Example.

1st equ. $5x-3y=90$; 2d equ. $2x+5y=160$,

whence $x=\dfrac{90+3y}{5}$, which value substituted, instead of

x, in the 2d equation, will give $\dfrac{180+6y}{5}+5y=160$

and $y=20$.

RULE III. *Multiply the first equation by the coefficient of* x *(or* y*) in the second equation; again, multiply the second equation by the coefficient of* x *(or* y*) in the first equation; then either subtract or add the new equations hence arising, according as the said coefficients have like or different signs, and you will get an equation with only one unknown quantity. The rest as before.*

 Ex-

Example.

1ſt equ. $2x + 5y = 160$ $\times 5$ 2d equ. $5x - 3y = 90$ $\times 2$

3d $10x + 25y = 800$ 4th $10x - 6y = 180$

4th $10x - 6y = 180$ $-$

$*\quad 31y = 620$, and $y = 20$.

Or thus:

1ſt equ. $2x + 5y = 160$ $\times 3$ 2d equ. $5x - 3y = 90$ $\times 5$

3d $6x + 15y = 480$ 4th $25x - 15y = 450$

4th $25x - 15y = 450$ $+$

$31x\quad * = 930$, and $x = 30$.

The ſame Example may be written as follows:

$$\begin{array}{ll} \text{1ſt equ.} & 2x + 5y = 160 \\ \text{2d} & 5x - 3y = 90 \end{array}\Big\} \times \begin{array}{l} 5 \\ 2 \end{array}\Big\{ \begin{array}{l} 10x + 25y = 800 \\ 10x - 6y = 180 \end{array} -$$

$$*\quad 31y = 620;\ y = 20.$$

Or thus:

$$\begin{array}{ll} \text{1ſt equ.} & 2x + 5y = 160 \\ \text{2d} & 5x - 3y = 90 \end{array}\Big\} \times \begin{array}{l} 3 \\ 5 \end{array}\Big\{ \begin{array}{l} 6x + 15y = 480 \\ 25x - 15y = 450 \end{array} +$$

$$31x\quad * = 930;\ x = 30.$$

RULE

RULE IV. *Take* S, *or* D *for the ſum or difference of the unknown quantities, then endeavour to get by ſubſtitution a final equation involving only the ſum or difference required, the value whereof found as uſual, and ſubſtituted in the other equations will give the values of the unknown quantities themſelves.*

Example I.

$$\begin{array}{lcl} 4x+7y=18, & \text{or} & 4S+3y=18 \\ 11x+8y=27, & & 8S+3x=27 \\ \hline 15x+15y=45, & & 12S+3S=45 \end{array}$$

Hence $15S=45$ and $S=3$

$$y=\frac{18-4S}{3}=\frac{18-12}{3}=2$$

$$x=\frac{27-8S}{3}=\frac{27-24}{3}=1.$$

Example II.

$$\begin{array}{lcl} 4x+7y=18, & \text{or} & 4S+3y=18 \\ 11x+14y=39, & & 11S+3y=39 \\ \hline 7x+7y=21 & & 7S \quad * \quad =21 \ \&\ S=3 \end{array}$$

Hence $y=\frac{18-4S}{3}=\frac{18-12}{3}=2$

$$x=\frac{18-7y}{4}=\frac{18-14}{4}=1.$$

Example III.

$$\begin{array}{ll} 7x-4y=10, & \\ 8x+19y=35, & \\ \hline 15x+15y=45, & + \end{array} \quad \text{or} \quad \begin{array}{l} 7x+7y-11y=10, \\ 8x+8y+11y=35, \\ \hline 15x+15y \quad * \quad =45 \quad + \end{array}$$

that is
$$\begin{array}{l} 7S-11y=10 \\ 8S+11y=35 \\ \hline 15S \quad * \quad =45 \quad + \end{array}$$

Whence $S=3$, $y=\frac{35-8S}{11}=1$, and $x=\frac{10+4y}{7}=2$.

Example IV.

$$\begin{array}{l} 7x-4y=10, \\ 16x+5y=37, \\ \hline 9x+9y=27, \end{array} - \quad \text{or} \quad - \begin{array}{l} 7x+7y-11y=10, \\ 5x+5y+11x=37, \\ \hline -2x-2y+11x+11y=27, \end{array}$$

that is
$$\begin{array}{l} 7S-11y=10 \\ 5S+11x=37 \\ \hline -2S+11S=27 \end{array} -$$

Whence $9S=27$, and $S=3$,

$y=\frac{7S-10}{11}=1$, and $x=\frac{37-5S}{11}=2$.

Example

Example V.

$$\begin{array}{rl} 7x - 4y = 10, & \\ 8x - 11y = 5, & \\ \hline 15x - 15y = 15, & + \end{array} \qquad \text{or} \qquad \begin{array}{rl} 4D + 3x = 10 & \\ 8D - 3y = 5 & \\ \hline 12D + 3D = 15 & + \end{array}$$

Hence $15D = 15$ and $D = 1$

$$x = \frac{10 - 4D}{3} = 2 \text{ and } y = \frac{8D - 5}{3} = 1$$

Example VI.

$$\begin{array}{rl} 7x - 4y = 10, & \\ 14x - 11y = 17, & \\ \hline 7x - 7y = 7, & - \end{array} \qquad \text{or} \qquad \begin{array}{rl} 4D + 3x = 10 & \\ 11D + 3x = 17 & \\ \hline 7D \quad * = 7, \& D = 1 & - \end{array}$$

Hence $x = \frac{10 - 4D}{3} = 2$

$$y = \frac{7x - 10}{4} = 1.$$

Example

Example VII.

$$\begin{array}{rl} 7x+\ 4y=18 & \\ 16x-27y=\ 5 & \\ \hline 23x-23y=23 & + \end{array} \quad \text{or} \quad \begin{array}{rl} 7x-\ 7y+11y=18 & \\ 16x-16y-11y=\ 5 & \\ \hline 23x-23y \quad * =23 & + \end{array}$$

$$\text{that is} \quad \begin{array}{rl} 7D+11y=18 & \\ 16D-11y=\ 5 & \\ \hline 23D \quad * =23 & + \end{array} \text{ and } D=1$$

Whence $y=\frac{18-7D}{11}=1$, and $x=\frac{18-4y}{7}=2.$

Example VIII.

$$\begin{array}{rl} 7x+4y=18, & \\ 16x-5y=27, & \\ \hline 9x-9y=\ 9, & - \end{array} \quad \text{or} \quad \begin{array}{rl} 7x-7y\ +\ 11y\ =\ 18, & \\ 5x-5y\ +\ 11x\ =\ 27, & \\ \hline -2x+2y+11x-11y=\ 9, & - \end{array}$$

$$\text{that is} \quad \begin{array}{rl} 7D+11y=18 & \\ 5D+11x=27 & \\ \hline -2D+11D=9 & - \end{array}$$

Whence $9D=9$ and $D=1$,

$$y=\frac{18-7D}{11}=1, \text{ and } x=\frac{27-5D}{11}=2.$$

Notes.

Notes. I. If the equations have any fractions, they muſt be previouſly cleared of them.

II. In applying of Rule III. if the coefficients of the unknown quantity which ought to be exterminated, admit of a common diviſor, it muſt be taken away by diviſion, and the quotients hence ariſing, uſed inſtead of the ſaid coefficients, will furniſh a more ſimple final equation.

Example.

$$\left.\begin{array}{r}20x+9y=49\\12x-7y=17\end{array}\right\}\times\begin{array}{r}3\\5\end{array}\left\{\begin{array}{l}60x+27y=147\\60x-35y=\ \ 85\end{array}\right.\text{becauſe}\begin{array}{r}20=4\times5\\12=4\times3\end{array}$$

$$-\quad 62y=\ 62 \text{ or } y=1.$$

III. The ſubſtitution of S (Rule IV.) in the equations involving three or more unknown quantities, gives ſometimes very ſimple reſolutions, as may be ſeen in the following Example, and in the Queſtions.

IV. When the mark of ſubtraction is put before the line drawn under the equations, as in Examples II. IV. VI. and VIII. of Rule IV. the ſuperior equation muſt be ſubtracted from the inferior; and that is only done in order to get a poſitive remainder in the ſubtraction of the numbers or known terms.

III. *Reſolution of Equations involving three or more unknown Quantities.*

RULE. *By one of the explained methods one of the unknown quantities may be exterminated from the given equations, then another from the new equations, and ſo on, till there remains a final equation, the reſolution of which will furniſh alſo the reſolution of the others.*

Ex-

Example.

$$\left.\begin{array}{r} x+\dfrac{y+z}{2}=34 \\ \dfrac{x}{3}+y+\dfrac{z}{3}=34 \\ \dfrac{x}{4}+\dfrac{y}{4}+z=34 \end{array}\right\} \begin{array}{c} 2 \\ \times 3 \\ 4 \end{array} \left\{\begin{array}{l} 2x+\ y+\ \ z=\ 68. \\ x+3y+\ \ z=102. \\ x+\ y+4z=136. \end{array}\right.$$

$$\text{Then } x=\frac{68-y-z}{2}$$
$$x=102-3y-z$$
$$x=136-y-4z$$

$$\text{Therefore } \frac{68-y-z}{2}=102-3y-z.$$
$$102-3y-z=136-y-4z.$$

$$\text{Whence } y=\frac{136-z}{5}.$$
$$y=\frac{3z-34}{2}.$$

$$\text{Consequently } \frac{3z-34}{2}=\frac{136-z}{5}.$$

Hence $z=26$, $y=22$, and $x=10$.

By

By the substitution of S.

$$\begin{aligned} 2x+\ y+\ z &= 68 \\ x+3y+\ z &= 102 \\ x+\ y+4z &= 136 \end{aligned}$$

that is $\left.\begin{aligned} x+S &= 68 \\ 2y+S &= 102 \\ 3z+S &= 136 \end{aligned}\right\} \times \begin{matrix} 6 \\ 3 \\ 2 \end{matrix} \left\{\begin{aligned} 6x+\ 6S &= 408 \\ 6y+\ 3S &= 306 \\ 6z+\ 2S &= 272 \end{aligned}\right.$

$$\overline{6S+11S=986}\ +$$

Hence $17\,S = 986$, $S = 58$, $x = 68 - S = 10$, $y = \frac{102-S}{2} = 22$, and $z = \frac{136-S}{3} = 26$.

SECTION II.

RESOLUTION OF EQUATIONS OF ALL ORDERS.

I. *Solution of Equations, whose Roots are commensurate.*

DEFINITION. The *absolute term* of a final equation is that into which the unknown quantity does not enter.

RULE. *All the terms of the final equation being brought to one side, find all the divisors of the* absolute term *and substitute one of them at pleasure in the place of of the unknown quantity in the equation. If the divisor substituted (which may be called the* first divisor) *gives a*

result

result $=0$, *it shall be a root of the equation; but if there is a remainder, which may be called* error, *find all the divisors of it, and either subtracting each of these divisors severally from the* first divisor *if it be too great, or adding them to it severally if it be too little, write down all the numbers hence arising, which may be called* corrections; *then mark with a little star all those corrections which are found to be equal to any divisor of the* absolute term, *because among them must be necessarily the roots required. Therefore substituting again one of these corrections instead of the unknown quantity in the equation, and repeating, if there is still an error, the said operations; the new corrections which are found to be equal to any of the foregoing corrections which are marked with the little stars may be the roots required, and they ought to be by the same method determined.*

Example I.

Let $xx=16x-60$, that is $xx-16x+60=0$.

Absolute term.	Its divisors.
	1
60	2
30	2, 4
15	3, 6, 12
5	5, 10, 15
1	20, 30, 60.

Trial.

Trial.	Error.	Its divisors.
		1
$x = 12$	12	2
$16 = 16$	6	2, 4
$\times$	3	3, 6, 12
$16x = 192$	1	
$x^2 = 144$		
$-$		
$x^2 - 16x = -48$		
$60 = 60$		1
$+$		
$x^2 - 16x + 60 = 12$		

Corrections.

$12 - 1 = 11$

$12 - 2 = 10$* a root.

$12 - 3 = 9$

$12 - 4 = 8$

$12 - 6 = 6$* a root.

$12 - 12 = 0$

Proof 1st.

$$\begin{array}{rr} x = & 10 \\ 16 = & 16 \\ \hline \times\quad 16x = & 160 \\ x^2 = & 100 \\ \hline -\quad x^2 - 16x = & -60 \\ 60 = & 60 \\ \hline +\quad x^2 - 16x + 60 = & 0 \end{array}$$

Proof 2d.

$$\begin{array}{rr} x = & 6 \\ 16 = & 16 \\ \hline \times\quad 16x = & 96 \\ xx = & 36 \\ \hline -\quad x^2 - 16x = & -60 \\ 60 = & 60 \\ \hline +\quad x^2 - 16x + 60 = & 0 \end{array}$$

Example II.

Let $x^3-2x^2-33x+90=0$.

Abſolute term.	Its diviſors.
	1
90	2
45	3, 6
15	3, 9, 18
5	5, 10, 15, 30
1	45, 90.

Trial.	Error.	Its diviſors.
		1
$x=2$	24	2
$\times 33$	12	2, 4
$33x=66$	6	2, 8
$2x^2=8$	3	3, 6, 12, 24.
+	1	
$2x^2+33x=74$		
$x^3=8$		
−		
$x^3-2x^2-33x=-66$		
$90=90$		
+		
$x^3-2x^2-33x+90=24$		

Cor-

Corrections.

$$\begin{array}{l}
2+\ 1=\ 3 \ * \\
2+\ 2=\ 4 \\
2+\ 3=\ 5 \ * \\
2+\ 4=\ 6 \ * \\
2+\ 6=\ 8 \\
2+\ 8=10 \ * \\
2+12=14 \\
2+24=26
\end{array}$$

Trial.

$$\begin{array}{rr}
x= & 6 \\
\times 33 & \\ \hline
33x= & 198 \\
2x^2= & 72 \\
+ & \\ \hline
2x^2+33x= & 270 \\
x^3= & 216 \\
- & \\ \hline
x^3-2x^2-33x= & -54 \\
90= & 90 \\
+ & \\ \hline
x^3-2x^2-33x+90= & 36
\end{array}$$

Error.	Its divisors.	Corrections.
	1	$6-1=5$ * a root,
36	2	$6-2=4$
18	2, 4	$6-3=3$ * a root.
9	3, 6, 12	$6-4=2$
3	3, 9, 18, 36.	$6-6=0$
1		

Proof 1st.

$$\begin{array}{rr}
x= & 5 \\
\times 33 & \\ \hline
33x= & 165 \\
2xx= & 50 \\
+ & \\ \hline
2xx+33x= & 215 \\
x^3= & 125 \\
- & \\ \hline
x^3-2x^2-33x= & -90 \\
90= & 90 \\
+ & \\ \hline
x^3-2x^2-33x+90= & 0
\end{array}$$

Proof

Proof 2d.

$$\begin{array}{r}
x = 3 \\
\times 33 \qquad\quad \\
\hline
33x = 99 \\
2xx = 18 \\
+ \qquad\qquad\qquad \\
\hline
2x^2 + 33x = 117 \\
x^3 = 27 \\
- \qquad\qquad\qquad \\
\hline
x^3 - 2x^2 - 33x = -90 \\
90 = 90 \\
+ \qquad\qquad\qquad \\
\hline
x^3 - 2x^2 - 33x + 90 = 0
\end{array}$$

Notes. I. In any equation, the terms being regularly arranged as in the preceding examples, there are as many positive roots as there are changes in the signs of the terms from + to −, and from − to +; and the remaining roots are negative. Thus, in the last equation $x^3-2x^2-33x+90=0$, two being the changes in the signs, two are its positive roots, and the third remaining is negative.

II. To find the negative roots of an equation, both the first divisor and the corrections must be with the sign −. Thus, if in the last equation you suppose $x=-9$, you will find the error -504, hence the corrections -8, -7, -6, &c. and the true root $=-6$.

Trial.

Trial.

$$
\begin{array}{r}
x = \quad -9 \\
\times 33 \overline{\qquad\qquad} \\
33x = -297 \\
2x^2 = \quad 162 \\
+ \overline{\qquad\qquad\qquad} \\
2x^2 + 33x = -135 \\
x^3 = -729 \\
- \overline{\qquad\qquad\qquad\qquad} \\
x^3 - 2x^2 - 33x = -594 \\
90 = \quad 90 \\
+ \overline{\qquad\qquad\qquad\qquad\qquad} \\
x^3 - 2x^2 - 33x + 90 = -504
\end{array}
$$

Error.	Its divisors.
	1
-504	2
-252	2, 4
-126	2, 8
-63	3, 6, 12, 24
-21	3, 9, 18, &c.
-7	7, 14, 21, &c.
-1	

Corrections.

$-9+1=-8$

$-9+2=-7$

$-9+3=-6$ * the true root.

$-9+4=-5$ *

$-9+6=-3$ *

$-9+7=-2$ *

$-9+9=\quad 0$

Proof.

$-216-72+198+90=0.$

III. If

III. If the higheſt power of the unknown quantity has any coefficient, it may be taken away by ſubſtituting for this unknown quantity another divided by that coefficient. Let the equation be $xx-\frac{5x}{6}+\frac{1}{6}=0$, that is $6xx-5x+1=0$. Suppoſe $x=\frac{y}{6}$, then by ſubſtitution you will have $\frac{6yy}{36}-\frac{5y}{6}+1=0$; or $\frac{yy}{6}-\frac{5y}{6}+1=0$, and conſequently $yy-5y+6=0$. Now, the roots of this new equation being 2 and 3, that is $y=2$, and $y=3$, we get $x=\frac{2}{6}=\frac{1}{3}$ and $x=\frac{3}{6}=\frac{1}{2}$.

II. *Solution of Equations whoſe Roots are incommenſurate.*

WHEN the roots of an equation are incommenſurate, but poſſible, we can find only an approximate value for them, but to any degree of exactneſs required. There are various rules for this purpoſe; one of the moſt ſimple is that of Sir Iſaac Newton, which ſhall now be explained.

Let the equation be $x^3-2x-5=0$.

Firſt. Find the neareſt integer to the root; in this caſe the root is between 2 and 3, for theſe numbers being inſerted for x, the one give a poſitive and the other a negative reſult. Either the number above the root, or that below it may be aſſumed as the firſt value; only it will be more convenient to take that which appears to be neareſt to the root, as will be manifeſt from the operation.

Second.

Second. Suppose $x=2+f$, and substitute this value of x in the equation.

$$\begin{aligned} x^3 &= 8+12f+6ff+f^3 \\ -2x &= -4-2f \\ -5 &= -5 \end{aligned}$$

$$x^3-2x-5=-1+10f+6ff+f^3=0.$$

As f is less than unity, its powers ff and f^3 may be neglected in this approximation, and $10f=1$, or $f=\frac{1}{10}=0.1$ nearly, therefore $x=2.1$ nearly.

Third. As $f=0.1$ nearly, let $f=.1+g$, and insert this value of f in the preceding equation.

$$\begin{aligned} f^3 &= 0.001+0.03g+0.3g^2+g^3 \\ 6f^2 &= 0.06+1.2g+6g^2 \\ 10f &= 1+10g \\ -1 &= -1 \end{aligned}$$

$$f^3+6ff+10f-1=0.061+11.23g+6.3g^2+g^3=0$$

and neglecting gg and g^3 as very small, $0.061+11.23g=0$, or $g=\frac{-0.061}{11.23}=-0.00543$; hence $f=0.1+g=0.1-0.00543=0.09457$ nearly, and $x=2.09457$ nearly.

Fourth. This operation may be continued to any length, as by supposing $g=-0.00543+h$, and so on, and the value of $x=2.09455147$ nearly.

Notes. I. By the first operation a nearer value of x may be found thus; since $f=0.1$ nearly, and $-1+10f+6ff+f^3=0$, we have $f=\frac{1}{10+6f+ff}$

$$=\frac{1}{10+0.6+0.01}=\frac{1}{10.61}=0.0942$$ nearly, and $x=2.0942$ nearly.

II. In the same manner may the root of a *pure* equation be found, and this gives an easy method of approximating to the roots of numbers which are not perfect powers. A *pure* equation is that which contains only one power of the unknown quantity, as $2x^2-36=x^2$, $x^3-64=0$, $2x^4-20=x^4+70$, &c. if there are different powers of the unknown quantity, the equation is called *affected*; thus $xx=16x-60$ is an affected quadratic equation, and $x^3-2x^2-33x+90=0$ is an affected cubic equation.

CHAPTER IV.

OF SPEAKING ARITHMETICO-ALGEBRAICALLY.

AN explanation of the analytical art in order to establish a general method of resolving questions, is the proper object treated of in this Chapter: and because the resolution of questions involves quantities with the sign + or — prefixed to them, which are called affirmative and negative, it is also necessary to explain the meaning

meaning of these different and contrary signs. Hence this chapter is naturally divided into the two following Sections.

SECTION I.

OF AFFIRMATIVE AND NEGATIVE QUANTITIES.

DEFINITIONS. An *affirmative* quantity is a quantity greater than nothing, and is known by this sign $+$: a *negative* quantity is a quantity less than nothing, and is known by this sign $-$: thus $+2$, or $+x$, mean that the quantities 2, or x, are affirmative; but -2, or $-x$, signify that the quantities 2, or x, are negative.

When quantities are considered abstractedly, then $+$ and $-$ denote Addition and Subtraction only, and the terms *positive* and *negative* express the same ideas. In that case a negative quantity by itself is unintelligible.

In questions, however, there may be an opposition and contrariety in the quantities, analogous to that of Addition and Subtraction; and the signs $+$ and $-$ may very conveniently be used to express that contrariety. In such cases, negative quantities are understood to exist by themselves; and the same rules take place in operations into which they enter, as are used with regard to the negative terms of abstract quantities.

The possibility of any quantity's being less than nothing is to some a very great paradox, if not a downright absurdity: and truly so it would be, if we should suppose it possible for a body or substance to be less than nothing: but quantities, whereby the different de-

grees of qualities are estimated, may be easily conceived to pass from affirmation through nothing into negation. Thus a person in his fortunes may be said to be worth 2000 pounds, or 1000, or nothing, or —1000, or —2000; in which two last cases he is said to be 1000 or 2000 pounds worse than nothing; thus a body may be said to have two degrees of heat, or one degree, or no degree, or — one degree, or — two degrees, &c. Certain it is, that all contrary quantities do necessarily admit of an intermediate state, which alike partakes of both extremes, and is best represented by a cypher or 0: and if it is proper to say that the degrees on either side this common limit are greater than nothing; I do not see why it should not be as proper to say on the other side, that the degrees are less than nothing, at least in comparison to the former. That which most perplexes narrow minds in this way of thinking is, that in common life most quantities lose their names when they cease to be affirmative, and acquire new ones as soon as they begin to be negative; thus we call negative goods, debts; negative gain, loss; negative heat, cold; negative descent, ascent, &c. and in this sense, indeed, it may not be so easy to conceive how a quantity can be less than nothing, that is, how a quantity under any particular denomination can be less than nothing, so long as it retains that denomination. But the question is, whether, of two contrary quantities under two different names, one quantity under one name may not be said to be less than nothing, when compared with the other quantity, though under a different name; whether any degree of cold may not be said to be further from any degree of heat, than lukewarmness, or no heat at all. Difficulties that arise from the imposition of scanty and limited names upon quantities which in themselves are actually unlimited, ought to be charged upon those names, and not upon the things themselves. When the quantities are abstractedly considered without any

any regard to the degrees of magnitude, the names of quantities are as extenſive as the quantities themſelves; ſo that all quantities that differ only in degree one from another, how contrary ſoever they may be one to another, paſs under the ſame name; and affirmative and negative quantities are only diſtinguiſhed by their ſigns, and not by their names, the ſame number or letter repreſenting both: theſe ſigns therefore in mathematical language carry the ſame diſtinction along with them, as do particles and adjectives ſometimes in common language, as in the words convenient and inconvenient, happy and unhappy, good health and bad health, &c.

Theſe affirmative and negative quantities, as they are contrary to one another in their own natures, ſo likewiſe are they in their effects; a conſideration which, if duly attended to, would remove all difficulties concerning the ſigns of quantities ariſing from Addition, Subtraction, Multiplication, Diviſion, &c. for the reſult of working by affirmative quantities in all theſe operations is known; and therefore like operations in negative quantities may be known by the rule of contraries.

Note. The ſign of a negative quantity is never omitted, nor the ſign of an affirmative one, except when ſuch an affirmative quantity is conſidered by itſelf, or happens to be the firſt in a ſeries of quantities ſucceeding one another: thus we do not often mention the quantity $+2$, or $+x$, but the quantity 2, or x; nor the ſeries $+2-3+x+5$, $+x-4+6-3$, &c. but the ſeries $2-3+x+5$, and $x-4+6-3$, &c.

SEC-

SECTION II.

GENERAL METHOD OF RESOLVING QUESTIONS.

THIS method confifts of fome rules concerning *Notation*, *Equation*, *Refolution*, *Anfwer*, and *Verification*. A diftinct conception of the nature of the queftion, and of the relations of the feveral quantities to which it refers, will generally lead to the proper method of ftating it, which in effect may be confidered only as a tranflation from common language into that of Analyfis. But, in order to enable my young fcholar to find out the folution required, I fhall give fome Rules which will eafily lead to the proper method of ftating a queftion and refolving it.

I. *Notation.*

Notation gives the proper names to the unknown quantities.

RULES. I. If any queftion, in which there are two or more quantities fought, can be refolved by means of one letter, *the leaft of them muft be reprefented arbitrarily by fome letter of the alphabet, and the reft receive their names from fo many conditions of the queftion, expreffing their relation to the faid leaft quantity.* Thus, the

the ſolution becomes more eaſy and elegant. (See Ch. V. Queſt. II. III. VIII. and XX. in the Note.)

II. If there are any fractions of the unknown quantity, *it ought to be marked with a letter and a numeral coefficient expreſſing the leaſt multiple of all the denominators.* Thus the fractions are avoided, and the ſolution is more ſimple. (See Ch. V. Queſt. I. IV. and VI.)

III. It is required to repreſent by letters, not only the quantities ſought, but alſo every unknown quantity of the queſtion, in order to get the equations, which are neceſſary to its ſolution. (See Chap. V. Queſt. VI. VII. IX. X. XI. XII. XIII. XIV. XV. XX. XXI. and Chap. VI. Queſt. I. II. III. IV.)

IV. *Sometimes it is convenient to expreſs by letters, not the unknown quantities themſelves, but ſome other quantities connected with them, as their ſum, difference, &c. from which they may be eaſily derived.* (See Chap. V. Queſt. XVIII. XIX. XX. XXIII. XXIV. and Chap. VI. Queſt. V. and VII.)

II. *Equation.*

The word *Equation* is taken here in a ſenſe different from the common, and ſignifies the method of drawing out from a queſtion the equations neceſſary to its ſolution.

RULES. I. *After having given to the quantities ſought the proper names, work them according to the nature of the queſtion, in the ſame manner as you worked the Falſe Poſitions in the Arithmetical Language; only with this*

this difference, that there you really performed upon the false numbers all the operations required by the conditions of the question, but here you must only mark with the requisite signs those operations which cannot be performed upon the letters. Every result compared with the like result given in the question will afford an equation. (See Quest. I. II. &c. XI. XII. &c. of Chap. V.)

II. *Every condition which is not made use of in the notation, if translated from the common language into that of our analysis, will furnish an equation.* (See Chap. V. Quest. XVI. XVII. and XX. in the Note.)

III. *If one and the same unknown quantity may be expressed by two different ways of notation, for instance, arbitrarily and by means of a condition; these two expressions connected by the sign of equality will give an equation.* (See Chapter V. Quest. II. V. and VI. and Chap. VI. Quest. I. II. III. and IV.)

Note. The learner must be very careful to make no use of the same condition twice, except in company with others that have not been considered. (See Chap. V. Quest. XXII.)

III. *Resolution.*

Resolution comprehends the deducing final equations and resolving them.

RULES. I. *When there is only one unknown quantity to be found, the final equation is deduced by the preceding method of drawing out from a question the equations necessary to its solution.* (II. Equation.)

II. *When*

II. *When there are two or more unknown quantities to be found, and all the equations are ſimple, the final equation is deduced by the rules explained in the firſt Section of Chapter* III.

III. *When there are two or more unknown quantities to be found, and the equations of a ſuperior order, every power of the unknown quantity to be exterminated, muſt be conſidered as an unknown quantity by itſelf, and then the final equation is deduced by the ſame rules belonging to the finding of ſimple final equations.* (See Chap. VI. Queſt. VI. and VII.)

IV. *When the final equations are ſimple, they are reſolved as ſimple equations containing only one unknown quantity.* (Chap. III. Sect. I.)

V. Laſtly, *when the final equations are of a ſuperior order, they are reſolved by the methods given in ſecond Section of Chapter* III.

Notes. I. In deducing final equations from the equations containing two or more unknown quantities, no equation muſt be left out of the account.

II. The method announced in the third Rule will be particularly explained in the Algebraical Language.

IV. *Anſwer.*

The *Anſwer* determines not only the value of every unknown quantity but alſo how many different values can be given to it, and what are poſſible, what impoſſible, what uſeful, what uſeleſs, &c.

RULES. I. *The resolution of a final equation gives the different values of the unknown quantity contained therein; and then by substitution the values of the other unknown quantities are determined* (See Chap. III. Sect. I. and Chap. V. Sect. II. and III.)

II. *Any equation admits of as many resolutions, or has as many roots as there are units in the exponent of its order.* Thus, a simple equation has only one root, a quadratic equation two, a cubic three, a biquadratic four, and so on, an equation of the 5th, 6th, &c. order has five, six, &c. roots.

III. *A root is possible and expressible, when its value may be exactly obtained.* Thus, the roots of the quadratic equation $xx+4x=96$ are possible and expressible, because they are $x=8$ and $x=-12$. (See Chap. III. Sect. II. of the commensurate roots.)

IV. *A root is possible and inexpressible when its value can be only obtained by approximation, but to any degree of exactness required.* Thus, the roots of the quadratic equation $xx-4x=-1$ are possible but inexpressible, because they can be only obtained by approximation, which carried, for instance, to three decimal places, gives $x=3.732$ and $x=0.268$, whence arises the very small error -0.000176, because substituting for x each of these roots, the given equation becomes, in both the cases, $xx-4x=-1.000176=-1$ very nearly. (See Chap. III. Sect. II. of the incommensurate roots, and Chap. VI. Quest. VI.)

V. *A root becomes impossible when it cannot be expressed either exactly or by approximation.* Thus the roots of the quadratic equation $xx-4x=-6$ are impossible, because their nearest value is $x=2$, which is neither exact nor

ca-

capable to be carried to any other approximate degree of exactness. (See Chap. III. Sect. II. of the Incommensurate Roots, and Chap. VI. Quest. V. VI. and VII.)

VI. *Every root is useful when it gives a just solution of the question, that is, when the assumed value of each unknown quantity, and therefore the solution itself, have their proper meaning*, as may appear from considering the conditions of the question.

VII *A root is useless when it gives no just answer to the question, that is, when the assumed value of any unknown quantity, and therefore the answer itself has no proper meaning*, as will appear by consulting the conditions of the question. (See Chap. VI. Quest. I. II. III. IV. and VII.)

Notes. I. Every root, either possible or impossible, expressible or inexpressible, useful or useless, gives an answer to the question, because its value being inserted would answer the conditions, that is, would correspond to the original equations, by which these conditions were expressed. This truth will be generally demonstrated in the Algebraical Language. (See Chap. VI. and VII.)

II. Any question admits of as many just answers as there are different useful values of the unknown quantities. (See Chap. VI. Quest. VI. and VII.)

III. The useful values of the unknown quantities are determined from the nature of the things represented by them, and from the conditions of the question.

IV. The

IV. The impoſſible values of the unknown quantities, and other properties of the roots of equations, will be explained and demonſtrated in the Algebraical Language.

V. *Verification.*

Verification is the demonſtration of the given anſwer.

RULE. *After having determined by the anſwer the numbers ſought, work them according to the conditions of the queſtion, and if the reſults give no error, the queſtion is reſolved and the anſwer itſelf demonſtrated.*

Notes. I. In the following determinate queſtions I put down the anſwer immediately after the queſtion, and then its ſolution or the method of reſolving it; becauſe this way of putting down the anſwer firſt, will not only ſerve to illuſtrate the following ſolution, but may alſo ſerve to fix the queſtion more firmly in the minds of young beginners, who are too apt to neglect it, and to ſubſtitute chimerical notions of their own, that are not to be juſtified, either from the conditions of the queſtion, or common ſenſe.

II. After the learner has run over ſome of theſe queſtions, and has got a tolerable inſight into the method of their ſolutions, it will be very proper for him to begin again, and to attempt the ſolution of every queſtion himſelf, and not to have recourſe to the ſolutions here given, but in caſes of abſolute neceſſity: but after the work is over, he may then compare his own ſolution with that which is here given, and may alter or reform it as he thinks fit.

CHAPTER V.

SOLUTION OF DETERMINATE QUESTIONS PRODUCING SIMPLE EQUATIONS.

FROM the reſolution of equations we obtain the reſolution of a variety of uſeful problems, both in pure mathematics and phyſics, and alſo in the practical arts founded upon theſe ſciences. In this place, we conſider the application of it to thoſe queſtions where the quantities are expreſſed by numbers, and their magnitude alone is to be conſidered.

SECTION I.

QUESTIONS INVOLVING ONLY ONE UNKNOWN QUANTITY.

QUESTION I. A ſchool maſter being asked how many ſcholars he had, ſaid, If I had as many, half as many, and one quarter as many more, I ſhould have 88.

Anſwer.

The ſcholars are 32, becauſe $32+32+\frac{32}{2}+\frac{32}{4}=64+16+8=88$.

So-

Solution.

Let the number of scholars be x or $4y$, to avoid the fractions. Therefore, in the 1st case

$$x+x+\frac{x}{2}+\frac{x}{4}=88$$

$$\times 4$$

$$4x+4x+2x+x=352$$

that is $11x=352$, and $x=\frac{352}{11}=32$.

Or in the 2d.

$$4y+4y+2y+y=88$$

that is $11y=88$, $y=\frac{88}{11}=8$ and $4y=32$.

Quest. II. One has six sons, each whereof is 4 years older than his next younger brother, and the eldest is three times as old as the youngest; what are their several ages?

Answer.

Their several ages are 30, 26, 22, 18, 14, and 10. For $30=26+4$; $26=22+4$; $22=18+4$; $18=14+4$; $14=10+4$; and $30=3\times 10$.

Solution.

Let the age of the youngest son be x

The age of the 5th shall be	$x+4$	1st cond.
4th	$x+8$	
3d	$x+12$	
2d	$x+16$	
1st	$x+20$	
1st	$3x$	2d cond.

Hence

Hence $3x = x + 20$

$\overline{} - x$

$2x = 20$

$\overline{} \div 2$

$x = 10.$

Quest. III. A, B, and C would divide 200*l.* between them, so that B may have 6*l.* more than A, and C 8*l.* more than B; how much must each have?

Answer.

A must have 60, B 66, and C 74; Because $60+66+74=200$; $66=60+6$, and $74=66+8$.

Solution.

Let the share of A be x
The share of B shall be $x+6$
C $x+6+8$

$\overline{} +$

Total $3x+20$

Hence $3x+20=200$

$\overline{} - 20$

$3x=180$

$\overline{} \div 3$

$x=60.$

Quest.

Queſt. IV. A trader allows 100*l.* per annum for the expences of his family, and augments yearly that part of his ſtock which is not ſo expended, by a third part of it; at the end of three years his original ſtock was doubled: what had he at firſt?

Anſwer.

He had 1480*l.* For

$$
\begin{array}{ll}
 & 1480 \\
 & \overline{} - 100 \\
 & 1380 \\
 & \overline{} \div 3 \\
 & 460 \\
 & \overline{} + \\
\text{1ſt year} & 1840 \\
 & \overline{} - 100 \\
 & 1740 \\
 & \overline{} \div 3 \\
 & 580 \\
 & \overline{} + \\
\text{2d year} & 2320 \\
 & \overline{} - 100 \\
 & 2220 \\
 & \overline{} \div 3 \\
 & 740 \\
 & \overline{} + \\
\text{3d year} & 2960 \\
 & \overline{} \div 2 \\
 & 1480 \text{ as at firſt.}
\end{array}
$$

Solution.

Solution.

Suppose he had $27x$. Then

$$
\begin{array}{ll}
 & 27x \\
 & \overline{\phantom{27x - \frac{2700}{27}}} - 100, \text{ or } - \frac{2700}{27} \\
 & \left\{\begin{array}{l} 27x - \frac{2700}{27} \\ \overline{\phantom{27x - \frac{2700}{27}}} \div 3 \\ 9x - \frac{900}{27} \end{array}\right. \\
 & \overline{\phantom{27x - \frac{2700}{27}}} + \\
\text{1st year} & 36x - \frac{3600}{27} \\
 & \overline{\phantom{27x - \frac{2700}{27}}} - 100, \text{ or } - \frac{2700}{27} \\
 & \left\{\begin{array}{l} 36x - \frac{6300}{27} \\ \overline{\phantom{27x - \frac{2700}{27}}} \div 3 \\ 12x - \frac{2100}{27} \end{array}\right. \\
 & \overline{\phantom{27x - \frac{2700}{27}}} + \\
\text{2d year} & 48x - \frac{8400}{27} \\
 & \overline{\phantom{27x - \frac{2700}{27}}} - 100, \text{ or } - \frac{2700}{27} \\
 & \left\{\begin{array}{l} 48x - \frac{11100}{27} \\ \overline{\phantom{27x - \frac{2700}{27}}} \div 3 \\ 16x - \frac{3700}{27} \end{array}\right. \\
 & \overline{\phantom{27x - \frac{2700}{27}}} + \\
\text{3d year} & 64x - \frac{14800}{27} = 54x \\
 & \overline{\phantom{64x - \frac{14800}{27} = 54x}} - 54x \\
 & 10x - \frac{14800}{27} = 0 \\
 & \overline{\phantom{10x - \frac{14800}{27} = 0}} \times \frac{27}{10} \\
 & 27x - 1480 = 0.
\end{array}
$$

Quest. V. A gentleman distributing money among some poor people, found he wanted 10*s.* to be able to give 5*s.* to each; therefore he gives each 4*s.* only, and finds he has 5*s.* left. To find the number of shillings and poor people.

Answer.

The number of poor is 15, and the number of shillings 65. For $15 \times 5 = 75 = 65 + 10$; and $15 \times 4 = 60 = 65 - 5$.

Solution.

Let the number of poor be x
The number of shillings will be $5x - 10$
The number of shillings is also $4x + 5$

$$\text{Hence } 5x - 10 = 4x + 5$$
$$\underline{\qquad\qquad - 4x + 10}$$
$$x = 15 \text{ and } 4x + 5 = 65$$

Quest. VI. A courier sets out from a certain place, and travels at the rate of 7 miles in 5 hours, and 8 hours after another sets out from the same place, and travels the same road at the rate of 5 miles in 3 hours: I demand how long, and how far, the first must travel before he is overtaken by the second?

Answer.

The first travelled 50 hours, the second 42, and the miles travelled by each are 70. For $5 : 7 :: 50 : 70$, and $3 : 5 :: 42 : 70$.

Solution.

Suppose the first travelled $15x$ hours.
Then the second travelled $15x - 8$.
Now $5 : 7 :: 15x : 21x$ miles of the 1st.
And $3 : 5 :: 15x - 8 : 25x - \frac{40}{3}$ miles of the 2d.

Hence

Hence $25x - \frac{40}{3} = 21x$

$$\underline{\qquad\qquad\qquad} - 21x + \tfrac{40}{3}$$

$$4x = \tfrac{40}{3}$$

$$\underline{\qquad\qquad} \times \tfrac{3}{4}$$

$$3x = 10$$

Therefore $15x = 5 \times 3x = 50$, $15x - 8 = 42$, and $7 \times 3x = 21x = 70$.

Quest. VII. There is a certain fishing rod consisting of two parts, whereof the upper part is to the lower as 5 to 7; and moreover, 9 times the upper part with 13 times the lower is equal to 11 times the whole rod and 36 inches over: I demand the length of the two parts.

Answer.

The upper part is 45 inches, and the lower 63; for $45 : 63 :: 5 : 7$, and $9 \times 45 + 13 \times 63 = 11 \times 108 + 36$, as will appear upon trial.

Solution.

Let the length of the upper part in inches be $= 5x$
The length of the lower will be $= 7x$

$$\underline{\qquad} +$$

And the length of the whole rod $= 12x$

$$\underline{\qquad\qquad} \times 11$$

$$132x$$

$$5x \times 9 = 45x$$
$$7x \times 13 = 91x$$
$$\underline{\qquad\qquad} +$$
$$136x$$

Hence $136x = 132x + 36$

$$\underline{\qquad\qquad\qquad} - 132x$$

$4x = 36$, $x = 9$, $5x = 45$, and $7x = 63$.

 Quest.

Quest. VIII. A certain person bought 2 horses, A and B, with the trappings, which cost 100*l.* now, if the trappings be laid on horse A, they made the horses of equal value; but if laid on the other B, they made him double the value of A: how much did the said horses cost?

Answer.

The prices of the horses A and B are respectively 200, and 300; for $200+100=300$, and $300+100=2\times 200$.

Solution.

Let the price of horse A be $=x$
The price of B will be $=x+100$

$$\text{Hence } x+100+100=2x, \text{ and } x=200.$$

Quest. IX. Two persons A and B were talking of their ages; says A to B, seven years ago I was just 3 times as old as you were, and 7 years hence I shall be just twice as old as you will be: I demand their present ages.

Answer.

A is 49 years old, and B 21; for since 7 years before A's age would be 42, and B's 14, it would be consequently $42=3\times 14$; and since 7 years after A's age would be 56, and B's 28, it would also be $56=2\times 28$.

Solution.

Solution.

Let $3x$ and x represent the ages of A and B respectively 7 years ago.

$3x+7$ and $x+7$ will represent respectively their present ages.

$3x+14$ and $x+14$ will represent respectively their ages 7 years hence.

Therefore $3x+14=2\times\overline{x+14}=2x+28$, and $x=14$.

Hence $3x+7=49$, A's present age.
$x+7=21$, B's present age.

Quest. X. One places a certain number of rods upright in a straight line, at equal distances one from another, the vacancies being no more than sufficient to contain two rods apiece, but finding that by this means his line would not reach above 125 inches, he extended it to 208 inches by opening the vacancies just as wide again as before; what was the number of rods?

Answer.

The number of rods were 84, and consequently the number of intervals 83; for if 2 rods admit of but one interval, 3 rods of 2, &c. 84 rods will admit of 83 intervals; which intervals, if they were to be filled, would take up 83×2, or 166 rods; therefore, if the first line had been full, it would have taken up $84+166$, or 250 rods; again, the number of rods sufficient to fill the vacancies of the second line was 83×4, or 332; therefore, if the second line had been full, it would have taken up $84+332$, or 416 rods: now, if this answer

anſwer be juſt, the lengths of theſe two lines ought to have the ſame proportion to one another, as have the numbers of rods they would have taken up had they been full, and ſo we ſhall find them; for 125 inches are to 208 inches, as 125×2 to 208×2, that is, as 250 rods to 416 rods, as was to be demonſtrated.

Solution.

Let the number of rods be x; then

The number of intervals will be $x-1$;

The number of rods in the firſt line, if filled, would take up $3x-2$;

The number of rods in the ſecond line, if filled, would take up $5x-4$.

Therefore the lines being as the numbers 125 and 208, we have $3x-2 : 5x-4 :: 125 : 208$.

Hence $\overline{5x-4} \times 125 = \overline{3x-2} \times 208$

That is $625x-500=624x-416$

Or, $625x-624x=500-416$, and $x=84$.

SECTION II.

QUESTIONS INVOLVING TWO UNKNOWN QUANTITIES.

IN the following determinate queſtions, I put down ſeparately the conſtituent parts of every ſolution, which are Notation, Equation, Reſolution, Anſwer, and Verification, in order to give a clear idea of the method of

of resolving questions, applying it strictly to the particular examples.

Quest. XI. One exchanges 6 French crowns and 2 French dollars for 45 shillings; and at another time 9 crowns and 5 dollars of the same coin for 76 shillings: I demand the distinct values of a crown and of a dollar.

Solution.

Notation. Let the value of a crown be x, and of a dollar be y.

The value of 6 crowns will be $6x$, and of 2 dollars $2y$.

The value of 9 crowns will be $9x$, and of 5 dollars $5y$.

Equation. $6x+2y=45$ by the 1st condition.
$9x+5y=76$ by the 2d condition.

Resolution. $3x+3y=31$ or $3S=31$ and $S=\frac{31}{3}$, taking $S=x+y$.

Hence $6x+2y=4x+2S=4x+\frac{62}{3}=45$, and $x=\frac{45}{4}-\frac{62}{12}=\frac{73}{12}$.

$6x+2y=6\times\frac{73}{12}+2y=\frac{438}{12}+2y=45$, and $y=\frac{45}{2}-\frac{219}{12}=\frac{51}{12}$.

Answer. The value of a crown is 73 pence or 6*s.* 1*d.*
The value of a dollar is 51 pence or 4*s.* 3*d.*

Verification.

	s.	*d.*		*s.*	*d.*
The value of 6 crowns is	36	6	and of 9 is	54	9
The value of 2 dollars is	8	6	and of 5 is	21	3
Total	45	0	Total	76	0

Quest.

Queſt. XII. Some young men and women had a reckoning of 37 crowns to pay for a feaſt, and theſe were their conditions; that every young man ſhould pay 3 crowns and every young woman 2: now, if there had been as many young men as young women, and as many young women as young men, the reckoning would have come to 4 crowns leſs than it did: how many young men and young women were there?

Solution.

Notation. Let the young men be x and the young women y.

Their reckoning would have been $3x+2y$ in the 1ſt caſe.
$3y+2x$ 2d

Equation.

$3x+2y=37$ or $x+2S=37$ by the 1ſt condition.
$2x+3y=33$ or $y+2S=33$ 2d

Reſol. ——————— +

$5x+5y=70$ or $5S=70$ and $S=14$.

Hence $x=37-2S=37-28=9$,
and $y=33-2S=33-28=5$.

Anſwer. The young men were 9 and the women 5.

Verification. The young men would have paid 27 *cr.* in the 1ſt caſe, and 15 *cr.* in the 2d.

The women would have paid 10 *cr.* in the 1ſt caſe, and 18 in the 2d.

	cr.		*cr.*
	27		15
	10		18
	—+		—+
Total	37	Total	33

Queſt.

Queſt. XIII. One lays out 2 ſhillings and 6 pence in apples and pears, buying his apples at 4 a penny, and his pears 5 a penny; and afterwards accommodates his neighbours with half his apples and one-third of his pears for 13 pence, which was the price he bought them for: I demand how many he bought of each ſort?

Solution.

Notation. Suppoſe he bought $8x$ apples and $15y$ pears. Then,

The price of the apples, $2x$; for $4 : 1 :: 8x : 2x$.
The price of the pears, $3y$; for $5 : 1 :: 15y : 3y$.

Equation.

$2x+3y=30d.$ or $y+2S=30$ by the 1ſt condition.
$x+y=13d.$ or $S=13$ by the 2d condition.

Reſolution. $y=30-2S=30-26=4$ and $x=13-y=13-4=9$.

Anſwer. He bought 8×9 or 72 apples, and 15×4 or 60 pears.

Verification.

The price of the apples is	$18d.$ and $\frac{1}{2}$ of it	$9d.$
The price of the pears is	$12d.$ and $\frac{1}{3}$ of it	$4d.$
	—+	—+
Total	$30d.$	$13d.$

Queſt. XIV. A jocky has 2 horſes A and B, whoſe values are ſought. He has alſo two ſaddles, one valued at 12*l.* the other at 2*l.* now if he ſets the better ſaddle upon A, and the worſe upon B, A will then be worth twice as much as B; but on the other hand, if he ſets the better ſaddle upon B, and the worſe upon A, B will then be worth 3 times as much as A. I demand the values of the horſes.

Solution.

Notation. Suppose x and y be the prices of the horses A and B.

The price of A with the better saddle is $x+12$, with the worse $x+2$.

The price of B with the worse saddle is $y+2$, with the better $y+12$.

Equation.

$x+12=2y+4$ or $2y-x=8$ by the 1st condition.
$y+12=3x+6$ or $y-3x=-6$ by the 2d condition.

Resolution. Take $D=y-x$, and the foregoing equations will be

$$\left.\begin{array}{r}D+\ y=\ \ 8\\ D-2x=-6\end{array}\right\}\text{or}\left\{\begin{array}{r}2D+2y=\ 16\\ D-2x=-6\end{array}\right.$$

$$\overline{}+$$

$$3D+2D=\ 10 \text{ and } D=2.$$

$$\text{Hence } y=8-D=6$$

$$x=\frac{D+6}{2}=4.$$

Answer. The prices of the horses A and B are 4 and 6.

Verification. The price of A with the better saddle is 16, with the worse 6.

The price of B with the worse saddle is 8, with the better 18.

Then the price 16 is twice the price 8.

And the price 18 is thrice the price 6.

Quest.

Quest. XV. A certain company at a tavern found, when they came to pay their reckoning, that if they had been 3 more in company to the same reckoning, they might have paid one shilling apiece less than they did; and that, had they been 2 fewer in company, they must have paid one shilling apiece more than they did: I demand the number of persons, and their quota.

Solution.

Notation. Let the number of persons be x, and what every one actually paid be y. Therefore their whole reckoning must have been xy. Now,

$$\begin{array}{rr}
\text{In the 1st supposition their number would be} & x+3 \\
\text{every one's particular reckoning} & y-1 \\
\hline
\text{their total reckoning} & xy+3y-x-3. \\
\text{In the 2d supposition their number would be} & x-2 \\
\text{every one's particular reckoning} & y+1 \\
\hline
\text{their total reckoning} & xy-2y+x-2.
\end{array}$$

Equation.

In the 1st case $xy+3y-x-3=xy$ or $3y-x-3=0$

In the 2d case $xy-2y+x-2=xy$ or $-2y+x-2=0$

Resolution. $y \quad * \quad -5=0$

and $y=5$.

Hence $x=2y+2=12$ and $xy=60$.

Answer. The number of persons were 12, what every one actually paid 5, and the whole reckoning 60.

Verification.

In the 1st case we have $\overline{12+3}\times\overline{5-1}=15\times4=60$.

In the 2d case we have $12-2\times5+1=10\times6=60$.

Quest. XVI. A man, his wife, and his ſon's ages, make up 96; ſo that the husband's and ſon's years make wife's +15, but the wife's and the ſon's together make the husband's +2. Quære their ages?

Solution.

Notation. Take x and y for the husband's and wife's ages; and by the 1ſt condition the ſon's age will be $96-x-y$.

Equation. *Reſolution.*

By the 2d condition $x+96-x-y=y+15$ or $96-y=y+15$, and $y=40\frac{1}{2}$.

By the 3d condition $y+96-x-y=x+2$ or $96-x=x+2$, and $x=47$.

Anſwer. The husband has 47 years, the wife $40\frac{1}{2}$, and the ſon $8\frac{1}{2}=96-87\frac{1}{2}$.

Verification. 1ſt condition $47+40\frac{1}{2}+8\frac{1}{2}=96$;
2d condition $47+8\frac{1}{2}=40\frac{1}{2}+15$;
3d condition $40\frac{1}{2}+8\frac{1}{2}=47+2$.

Queſt. XVII. Three merchants met at an inn, and found the ſum of their gains 780*l.* if you add the gain of the 1ſt and 2d, and from the ſum ſubtract the gain of the 3d, there remains the gain of the 1ſt + 82; but if you add the gains of the 2d and 3d, and from the ſum ſubtract the gain of the 1ſt, there remains the gain of the 3d − 43. Quære each man's ſhare?

Solution.

Notation. If the ſhares of the 1ſt and 2d merchants be x and y, the ſhare of the 3d will be $780-x-y$ by the 1ſt condition.

Equation.

$x+y-780+x+y=x+82$, 1ſt cond.

$y+780-x-y-x=780-x-y-43$, 2d cond.

That is $\left\{\begin{array}{r} x+2y=862 \\ -x+y=-43 \end{array}\right\}$ by Reduction.

Reſolution. $3y=819$ and $y=273$.

Hence $x=y+43=316$, and $780-x-y=191$.

Anſwer. The 1ſt merchant gained 316*l.* the 2d 273*l.* and the 3d 191*l.*

Verification. 1ſt condition $316+273+191=780$;
2d condition $316+273-191=316+82$;
2d condition $273+191-316=191-43$.

SEC.

SECTION III.

QUESTIONS INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

Quest. XVIII. Three persons A, B, C, were talking of their money; says A to B and C, give me half of your money, and I shall have 34; says B to A and C, give me a third part of your money, and I shall have 34; says C to A and B, give me a fourth part of your money, and I shall have 34. How much money had each?

Solution.

Notation. Put x, y, and z, for the three persons' money, and take $S=x+y+z$. It will be $S-x=y+z$, $S-y=x+z$, and $S-z=x+y$.

Equation.

By the 1st condition $x+\frac{S-x}{2}=34$,

By the 2d condition $y+\frac{S-y}{3}=34$,

By the 3d condition $z+\frac{S-z}{4}=34$,

That is, by Reduction,

$x+S$

$$\left.\begin{array}{r} x+S=2.34 \\ 2y+S=3.34 \\ 3z+S=4.34 \end{array}\right\} \times \begin{array}{r} 6 \\ 3 \\ 2 \end{array}\left\{\begin{array}{l} 6x+6S=12.34 \\ 6y+3S=\ \ 9.34 \\ 6z+2S=\ \ 8.34 \end{array}\right.$$

$$\overline{6S+11S=29.34.}\ +$$

Resolution.

Hence $S=58$, because $S=\frac{29.34}{17}$ for $17S$ or $6S+11S=29.34$.

Therefore $x=2.34-58=10$; $y=\frac{3.34-58}{2}=22$; and $z=\frac{4.34-58}{3}=26$.

Answer. A had 10, B 22, and C 26. For

Verification.

1st condition, $10+\frac{22+26}{2}=34$;

2d condition, $22+\frac{10+26}{3}=34$;

3d condition, $26+\frac{10+22}{4}=34$.

Quest. XIX. Three men have each such a sum of money, that if the 1st and 2d man's money be added to $\frac{1}{4}$ of what the 3d man has, that sum will be 92*l.* and if the 2d and 3d man's money be added to $\frac{1}{7}$ of the 1st man's money, that sum will be 92*l.* lastly, if $\frac{1}{4}$ of the 2d man's money be added to the 1st and 3d man's money, that sum will be 92*l.* also. Quære, each man's money.

Solution.

Solution.

Notation. Take the same denominations as in the last foregoing Question.

Equation.

By the 1st condition, $S-z+\frac{z}{2}=92,$

By the 2d condition, $S-x+\frac{x}{3}=92,$

By the 3d condition, $S-y+\frac{y}{4}=92,$

That is, by Reduction,

$$\left.\begin{array}{l} 2S-\ \ z=2.92 \\ 3S-2x=3.92 \\ 4S-3y=4.92 \end{array}\right\} \times \begin{array}{c} 6 \\ 3 \\ 2 \end{array}\left\{\begin{array}{l} 12S-6z=12.92 \\ 9S-6x=\ 9.92 \\ 8S-6y=\ 8.92 \end{array}\right.$$

Resolution.

$$29S-6S=29.92$$

Hence $S=116$, because $S=\frac{29.92}{23}$ for $23S$ or $29S-6S=29.92$.

$$z=2S-2.92=2\times\overline{S-92}=2\times\overline{116-92}=2\times24=48.$$

$$x=\frac{3S-3.92}{2}=\frac{3\times\overline{S-92}}{2}=\frac{3\times\overline{116-92}}{2}=\frac{3\times24}{2}=36.$$

$$y=\frac{4S-4.92}{3}=\frac{4\times\overline{S-92}}{3}=\frac{4\times\overline{116-92}}{3}=\frac{4\times24}{3}=32.$$

Answer. The 1st man had 36*l.* the 2d 32*l.* and the 3d 48*l.* For,

Verification. 1st condition, $36+32+\frac{48}{2}=92;$

2d condition, $32+48+\frac{36}{3}=92;$

3d condition, $36+48+\frac{32}{4}=92.$

Quest.

Quest. XX. To find a number consisting of three places, whose digits are in arithmetical proportion; if this number be divided by the sum of its digits, the quotient will be 48; and if from the number be subtracted 198, the digits will be inverted.

Solution.

Notation.

Let the three digits be x, y, z, and $S = x+y+z$.

Then the number is $100x+10y+z$, or $99x+9y+S$;

If the digits be inverted, $100z+10y+x$, or $99z+9y+S$.

Equation.

By the 1st cond. $x+z=2y$, or $x+z+y=2y+y$, that is $S=3y$.

By the 2d cond. $\frac{99x+9y+S}{S}=48$, or $\frac{99x+9y+3y}{3y}=48$, that is $3x=4y$.

By the 3d cond. $99x+9y+S-198=99z+9y+S$, that is $x-2=z$.

Resolution.

$$\begin{array}{r} x+z=2y \\ x-2=\ z \\ \hline 2x-2=2y \end{array}+ \qquad \text{Therefore } \begin{array}{r} 4x-4=4y \\ 3x\quad\ =4y \\ \hline x-4=0 \end{array}- \qquad \text{Hence } \begin{array}{r} x=4 \\ y=3 \\ \\ z=2 \end{array}$$

Answer.

The number sought is 432. For

Verification.

1st cond. $4-3=3-2$; 2d cond. $\frac{432}{9}=48$; 3d cond. $432-198=234$.

Note. This queſtion may be reſolved with only two unknown quantities, as follows.

Notation.

Let the three digits be x, $x+d$, $x+2d$, by the 1ſt condition.

Then the number is $100x+10x+10d+x+2d$, or $111x+12d$;

If the digits be inverted, it is $100x+200d+10x+10d+x$, or $111x+210d$.

Equation.

By the 2d cond. $\frac{111x+12d}{3x+3d}=48$, or $\frac{37x+4d}{x+d}=48$, that is $x+4d=0$.

By the 3d cond. $111x+12d-198=111x+210d$, that is $1+d=0$.

Reſolution.

$d=-1$, $x=-4d=-4\times-1=4$, $x+d=4-1=3$, $x+2d=4-2=2$.

Queſt. XXI. Suppoſe a dog, a wolf, and a lion, together, could devour a ſheep in 10 minutes, the dog and wolf in 20, and the wolf and lion in 12; the queſtion is, in what time each of them ſeparately would eat up the ſheep?

Solution.

Notation.

Let the times required for the dog, wolf, and lion, be respectively x, y, and z.

Then, because the parts of the sheep devoured are proportional to the times, it will be

$x : 1 :: 10 : \frac{10}{x} =$ to the part devoured by the dog in 10 minutes.

$y : 1 :: 10 : \frac{10}{y} =$ to the part devoured by the wolf in 10 minutes.

$z : 1 :: 10 : \frac{10}{z} =$ to the part devoured by the lion in 10 minutes.

$x : 1 :: 20 : \frac{20}{x} =$ to the part devoured by the dog in 20 minutes.

$y : 1 :: 20 : \frac{20}{y} =$ to the part devoured by the wolf in 20 minutes.

$y : 1 :: 12 : \frac{12}{y} =$ to the part devoured by the wolf in 12 minutes.

$z : 1 :: 12 : \frac{12}{z}$ — to the part devoured by the lion in 12 minutes.

The sheep being the same in all cases, is here represented by unity.

Hence we draw the following equations.

Equation.

By the 1st cond. $\frac{10}{x}+\frac{10}{y}+\frac{10}{z}=1$, or $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10}=\frac{6}{60}$.

By the 2d cond. $\frac{20}{x}+\frac{20}{y}=1$, or $\frac{1}{x}+\frac{1}{y}=\frac{1}{20}=\frac{3}{60}$.

By the 3d cond. $\frac{12}{y}+\frac{12}{z}=1$, or $\frac{1}{y}+\frac{1}{z}=\frac{1}{12}=\frac{5}{60}$.

Resolution.

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{6}{60}$$

$$\frac{1}{x}+\frac{1}{y}\qquad=\frac{3}{60}$$

$$*\quad *\quad \frac{1}{z}=\frac{3}{60}=\frac{1}{20} \text{ and } z=20.$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{6}{60}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{5}{60}$$

$$\frac{1}{x}\quad *\quad *=\frac{1}{60} \text{ and } x=60.$$

Hence $\frac{1}{y}=\frac{5}{60}-\frac{1}{z}=\frac{5}{60}-\frac{3}{60}=\frac{2}{60}=\frac{1}{30}$ and $y=30$.

Answer. The dog would devour the sheep in 60 min. the wolf in 30, and the lion in 20.

Veri-

Verification.

$$\left.\begin{array}{l} 60 : 1 :: 10 : \frac{10}{60} = \frac{1}{6} = \\ 30 : 1 :: 10 : \frac{10}{30} = \frac{2}{6} = \\ 20 : 1 :: 10 : \frac{10}{20} = \frac{3}{6} = \\ 60 : 1 :: 20 : \frac{20}{60} = \frac{2}{6} = \\ 30 : 1 :: 20 : \frac{20}{30} = \frac{4}{6} = \\ 30 : 1 :: 12 : \frac{12}{30} = \frac{2}{5} = \\ 20 : 1 :: 12 : \frac{12}{20} = \frac{3}{5} = \end{array}\right\} \begin{array}{l}\text{to the part}\\ \text{devoured by}\\ \text{the}\end{array} \left\{\begin{array}{l} \left.\begin{array}{l}\text{dog}\\ \text{wolf}\\ \text{lion}\end{array}\right\} \text{in 10 minutes.} \\ \left.\begin{array}{l}\text{dog}\\ \text{wolf}\end{array}\right\} \text{in 20 minutes.} \\ \left.\begin{array}{l}\text{wolf}\\ \text{lion}\end{array}\right\} \text{in 12 minutes.} \end{array}\right.$$

Whence $\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=1$, $\frac{2}{6}+\frac{4}{6}=1$, $\frac{2}{5}+\frac{3}{5}=1$.

Quest.

Queſt. XXII. Four gameſters A, B, C, and D, each with a different ſtock of money about him, but as yet unknown, ſit down to game; during their play A wins half of B's ſtock, B wins a third part of C's, C a fourth part of D's, and D a fifth part of A's; after this they all riſe with the ſame ſum of money about them, to wit, 23. It is required to determine the ſeveral ſtocks of money, with which the gameſters A, B, C, and D, began to play.

Solution.

Notation. Let the ſeveral ſtocks of money be a, b, c, d.

Equation.

$$\left.\begin{array}{ll} \text{By the 4th and 1ſt cond.} & \frac{4a}{5}+\frac{b}{2}=23 \\ \text{By the 1ſt and 2d cond.} & \frac{b}{2}+\frac{c}{3}=23 \\ \text{By the 2d and 3d cond.} & \frac{2c}{3}+\frac{d}{4}=23 \\ \text{By the 3d and 4th cond.} & \frac{3d}{4}+\frac{a}{5}=23 \end{array}\right\} \times \begin{array}{c} 6 \\ 6 \\ 3 \\ 1 \end{array} \left\{\begin{array}{l} \frac{24a}{5}+3b=6\times 23 \text{ equation 1ſt.} \\ 3b+2c=6\times 23 \text{ equation 2d.} \\ 2c+\frac{3d}{4}=3\times 23 \text{ equation 3d.} \\ \frac{3d}{4}+\frac{a}{5}=1\times 23 \text{ equation 4th.} \end{array}\right.$$

Re-

Resolution.

$$\frac{24a}{5}+3b+2c+\frac{3d}{4}=9\times 23 \text{ by adding equations 1st\&3d.}$$

$$\frac{a}{5}+3b+2c+\frac{3d}{4}=7\times 23 \text{ by adding equations 2d\&4th.}$$

$$\frac{23a}{5} \quad * \quad * \quad * =2\times 23, \text{ and } a=\frac{2\times 23\times 5}{23}=10.$$

$$\text{Hence } \frac{b}{2}=23-\frac{4a}{5}=23-\ 8=15 \text{ and } b=30$$

$$\frac{c}{3}=23-\frac{b}{2}=23-15=\ 8 \text{ and } c=24$$

$$\frac{d}{4}=23-\frac{2c}{3}=23-16=\ 7 \text{ and } d=28.$$

Answer.

The stocks required are 10, 30, 24, 28, belonging respectively to the four gamesters A, B, C, and D. For,

Verification.

$10-\ 2+15=23$, according to conditions 4 and 1.
$30-15+\ 8=23$, according to conditions 1 and 2.
$24-\ 8+\ 7=23$, according to conditions 2 and 3.
$28-\ 7+\ 2=23$, according to conditions 3 and 4.

Quest. XXIII. Four persons A, B, C, and D, owe a certain sum of money, so that A, B, and C together owe 210 crowns; A, B, and D 290; A, C, and D 300; B, C, and D 310: what did each of them owe?

Solution.

Solution.

Notation.

Let the ſeveral debts required be reſpectively A, B, C, and D; then take $S = A + B + C + D$.

Equation.

By the 1ſt cond. $A + B + C = 210$,
By the 2d cond. $A + B + D = 290$,
By the 3d cond. $A + C + D = 300$,
By the 4th cond. $B + C + D = 310$,

$$3A + 3B + 3C + 3D = 1110,$$

That is by ſubſtitution,

$$S - D = 210 \text{ and } D = S - 210.$$
$$S - C = 290 \text{ and } C = S - 290.$$
$$S - B = 300 \text{ and } B = S - 300.$$
$$S - A = 310 \text{ and } A = S - 310.$$

Reſolut. $4S - S = 1110$ and $S = 370$.

Hence $A = S - 310 = 60$, $B = S - 300 = 70$,
$C = S - 290 = 80$, $D = S - 210 = 160$.

Anſwer.

The ſeveral debts required are 60, 70, 80, and 160. For,

Verification.

$60 + 70 + 80 = 210$, according to the 1ſt condition,
$60 + 70 + 160 = 290$, according to the 2d condition,
$60 + 80 + 160 = 300$, according to the 3d condition,
$70 + 80 + 160 = 310$, according to the 4th condition.

Queſt.

Quest. XXIV. Five men have a mind to purchase a house valued at 2050*l.* says A to the others, if you give me $\frac{6}{7}$ of your money I can purchase the house alone; says B to the others, if you give me $\frac{7}{8}$ of your money, I shall be able to purchase the house; and so said C, D, and E, by turns, asking respectively $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, of the money of the others. How much money had each of them?

Solution.

Notation.

Put A, B, C, D, and E, for the several sums of money required, and take $S=A+B+C+D+E$.

Equation.

By the 1st condition, $A+\frac{6}{7}\times\overline{S-A}=2050$,

2d $B+\frac{7}{8}\times\overline{S-B}=2050$,

3d $C+\frac{8}{9}\times\overline{S-C}=2050$,

4th $D+\frac{9}{10}\times\overline{S-D}=2050$,

5th $E+\frac{10}{11}\times\overline{S-E}=2050$,

That is, by substitution,

$$\begin{array}{r}
A+\ 6S=\ 7\times 2050\\
B+\ 7S=\ 8\times 2050\\
C+\ 8S=\ 9\times 2050\\
D+\ 9S=10\times 2050\\
E+10S=11\times 2050\\
\hline
\end{array}+$$

Resolution.

$$41S \text{ or } S+40S=45\times 2050.$$

Therefore $S=\frac{45\cdot 2050}{41}=2250$.

Hence $A=7\times 2050-6S=850$,
$B=\ 8\times 2050-7S=650$, $C=\ 9\times 2050-8S=450$,
$D=10\times 2050-9S=250$, $E=11\times 2050-10S=50$.

 Answer.

Anfwer.
The money required are 850, 650, 450, 250, and 50.

Verification.

$850 + \frac{6}{7} \times 1400 = 850 + 1200 = 2050$, condit. 1ft.
$650 + \frac{7}{8} \times 1600 = 650 + 1400 = 2050$, condit. 2d.
$450 + \frac{8}{9} \times 1800 = 450 + 1600 = 2050$, condit. 3d.
$250 + \frac{9}{10} \times 2000 = 250 + 1800 = 2050$, condit. 4th.
$50 + \frac{10}{11} \times 2200 = 50 + 2000 = 2050$, condit. 5th.

CHAPTER VI.

SOLUTION OF DETERMINATE QUESTIONS PRODUCING EQUATIONS OF HIGHER ORDERS.

QUESTION I. One buys a certain number of oxen for 80 guineas, where it muft be obferved, that if he had bought 4 more for the fame money, they would have come to him a guinea apiece cheaper: what was the number of oxen?

Solution.

Notation. Let the number of oxen be x. Their coft being 80 guineas,

The price of one muft be $\frac{80}{x}$, for $x : 80 :: 1 : \frac{80}{x}$.

But if the number of oxen be $x+4$, then their coft being alfo 80 guineas,

The

The price of 1 should be $\frac{80}{x+4}$, for $x+4:80::1:\frac{80}{x+4}$.

This price is also $\frac{80}{x}-1$, because by supposition it is a guinea cheaper than the true price $\frac{80}{x}$. Therefore

Equation.

By supposition $\frac{80}{x+4}=\frac{80}{x}-1$,

or $80x=80x+320-x^2-4x$, that is $x^2+4x-320=0$.

Resolution.

The divisors of 320 are 1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 64, 80, 160, 320.

Trial.	Corrections.
$x=20$	$20-1=19$
$\times 4$	$20-2=18$
$4x=80$	$20-4=16$ * a root.
$x^2=400$	$20-5=15$
$+$	$20-8=12$
$x^2+4x=480$	$20-10=10$ *
$320=320$	$20-16=4$ *
$-$	$20-20=0$.
$x^2+4x-320=160.$	

The divisors of 160 are 1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80, 160.

Proof.

$$x=16$$
$$\times 4$$
$$4x=64$$
$$x^2=256$$
$$+$$
$$x^2+4x=320$$
$$-320$$
$$x^2+4x-320=0.$$

Answer.

Answer. The number of oxen is 16. The other root, -20, has no place in this question, a negative number being here unintelligible. See Notes I. and II. of Sect. II. N. I. of Chap. III. and Sect. I. of Chapter IV.

Verification.
$16 : 80 :: 1 : 5$, $20 : 80 :: 1 : 4$, and $4 = 5 - 1$.

Quest. II. A certain company at a tavern had a reckoning of 7*l*. 4*s*. to pay, upon which two of the company sneaking off, obliged the rest to pay 1*s*. apiece more than they should have done; what was the number of persons?

Solution.

Notation.
Let the number of persons be x. Their reckoning being 144*s*.

Their quota should have been $\frac{144}{x}$, for $x : 144 :: 1 : \frac{144}{x}$.

But the number of those who paid, being only $x-2$ by supposition,

Their quota must be $\frac{144}{x}+1$, and also $\frac{144}{x-2}$,

for $x-2 : 144 :: 1 : \frac{144}{x-2}$.

Equation.
Therefore $\frac{144}{x}+1=\frac{144}{x-2}$, or $x^2-2x+144x-288 = 144x$, that is $x^2-2x-288=0$.

Resolution. The divisors of 288 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, &c.

Trial.

Trial.	Corrections.
$x=24$ $\times 2$	$24-1=23$
$2x=48$	$24-2=22$
$x^2=576$	$24-3=21$
$x^2-2x=528$	$24-4=20$
$288=288$	$24-5=19$
$x^2-2x-288=240.$	$24-6=18$ * a root.

The divisors of 240 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, &c.

Proof.

$$\begin{array}{r} x=\ 18 \\ \hline \times 2 \\ 2x=\ 36 \\ x^2=324 \\ \hline x^2-2x=288 \\ 288=288 \\ \hline x^2-2x-288=\ 0. \end{array}$$

Answer. The persons were 18. The negative root, -16, must be neglected.

Verification. $18 : 144 :: 1 : 8$; $16 : 144 :: 1 : 9$; and $8+1=9$.

Quest. III. One lays out a certain sum of money in goods, which he sold again for 24*l.* and gained as much per cent. as the goods cost him. I demand what they cost him.

Solution.

Solution.

Notation. Put x for the money laid out. Then, becaufe 24 is the fum of the money laid out and its gain, this gain will be $24-x$. But the gains being as the principals, this gain is alfo $xx \div 100$, for $100 : x :: x : xx \div 100$. Therefore,

Equation.

$xx \div 100 = 24 - x$, that is $xx + 100x - 2400 = 0$.

Refolution. The divifors of 2400 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 25, 30, &c.

Trial.	Corrections.
$x = 25$	$25 - 1 = 24$ *
$\times 100$	$25 - 5 = 20$ * a root.
$100x = 2500$	$25 - 25 = 0$
$x^2 = 625$	

$$\begin{array}{r} x = 25 \\ \hline 100x = 2500 \quad \times 100 \\ x^2 = 625 \\ \hline x^2 + 100x = 3125 \quad + \\ 2400 = 2400 \\ \hline x^2 + 100x - 2400 = 725 \quad - \end{array}$$

$x^2 + 100x - 2400 = 725$, whofe divifors 1, 5, 25, &c.

Proof.

$$\begin{array}{r} x = 20 \\ \hline 100x = 2000 \quad \times 100 \\ x^2 = 400 \\ \hline x^2 + 100x = 2400 \quad + \\ 2400 = 2400 \\ \hline x^2 + 100x - 2400 = 0 \quad - \end{array}$$

Anfwer. The money laid out is 20*l.* The negative root, -120, has no meaning in our cafe, and therefore is neglected.

Veri-

Verification. The gain is $24-20$, that is 4, and therefore the gain per 100 is 20, as the question requires, because we find $20 : 4 :: 100 : 20$.

Quest. IV. One lays out $33l.\ 15s.$ in cloth, which he sold again for $48s.$ per piece, and gained as much in the whole as a single piece cost him. I demand how he bought in his cloth per piece.

Solution.

Notation. If the number of shillings, every single piece was bought for, be x,

The gain per piece will be $48 - x$ shillings. Hence

The whole gain will be $\frac{675 \times \overline{48-x}}{x}$ shills. because

$x : 48-x :: 675 : \frac{675 \times \overline{48-x}}{x}$, and $33l.\ 15s. = 675s.$

But the whole gain is also $= x$. Therefore

Equation.

$x = \frac{675 \times \overline{48-x}}{x}$, and hence $x^2 + 675x - 32400 = 0$.

Resolution. The divisors of 32400 are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 36, 40, 45, 48, &c.

Trial.	Corrections.
$x = 25$	$25+1=26$
$\times 675$	$25+2=27$ *
$675x = 16875$	$25+4=29$
$x^2 = 625$	$25+5=30$ *
$+$	$25+10=35$
$x^2+675x = 17500$	$25+20=45$ * a root.
$32400 = 32400$	

$x^2+675x-32400 = -14900$, whose divisors are 1, 2, 4, 5, 10, 20, 25, &c.

Proof.

Proof.

$$\begin{array}{r} x = \quad 45 \\ \hline 675x = 30375 \\ x^2 = \ \ 2025 \\ \hline x^2 + 675x = 32400 \\ 32400 = 32400 \\ \hline x^2 + 675x - 32400 = \qquad 0. \end{array} \begin{array}{l} \\ \times 675 \\ \\ \\ + \\ \\ \\ - \end{array}$$

Answer. Every single piece of the cloth costs 45*s*. the other root, -720, is of no use.

Verification. The gain per piece is $48-45$ or 3*s*. Hence $45 : 3 :: 675 : 45$, which is the whole gain, as in the question.

Quest. V. It is required to divide 16*l*. between two persons, so that the cube of the one's share may exceed the cube of the other's by 386.

Solution.

Notation. If the difference between the two shares be $2x$, the greater share will be $8+x$, and the less $8-x$. See page 22.

Equation. By the question $(8+x)^3-(8-x)^3=386$, or $x^3+192x-193=0$.

Resolution. The divisors of 193 are but 1 and 193. Hence $x=1$.

Answer. The two shares then are 9 and 7. The other roots are impossible.

Verification. $9+7=16$, 9^3-7^3 or $729-343=386$.

Quest. VI. What two numbers are those, whose product multiplied by the greater will produce 405, and their difference multiplied by the less 20?

Solution.

Notation. Let the greater number be x, and the less y.

Equation.
Then, by the question,

$$\begin{cases} xy \times x, \text{ or } x^2y = 405, \text{ and } x^2 = \dfrac{405}{y} \\ \overline{x-y} \times y, \text{ or } xy - yy = 20, \text{ and } x = \dfrac{20+yy}{y} \end{cases}$$

Resolution.
Hence

$\left(\dfrac{20+yy}{y}\right)^2 = \dfrac{405}{y}$, that is $y^4 + 40y^2 - 405y + 400 = 0$.

The divisors of 400 are 1, 2, 4, 5, 8, 10, 16, 20, 25, &c.

Trial.

$$\begin{array}{r} y = 1 \\ \hline \times 405 \\ 405y = 405 \\ 400 = 400 \\ \hline -405y + 400 = -5 \\ y^4 + 40y^2 = 41 \\ \hline + \\ y^4 + 40y^2 - 405y + 400 = 36, \end{array}$$

whose divisors are 1, 2, 3, 4, 6, 9, 12, 18, 36.

Corrections.	Proof.
$1+1=2$ *	$y= \quad 5$
$1+2=3$	$\times 405$
$1+3=4$ *	$405y=2025$
$1+4=5$ * a root.	$400= \quad 400$

$$\begin{array}{r} -405y+400=-1625 \\ y^4+40y^2= \quad 1625 \\ \hline y^4+40y^2-405y+400= \quad 0. \end{array}$$

Answer.

The less number is 5, and the greater 9, because $x=\frac{20+yy}{y}=\frac{20+25}{5}=\frac{45}{5}=9$.

Verification. Their product is 45, and $45\times 9=405$; their difference is 4, and $4\times 5=20$.

Note. Being $y^4+40y^2-405y+400=0$, and $y=5$ or $y-5=0$, we find $\frac{y^4+40y^2-405y+400}{y-5}=$ $y^3+5y^2+65y-80=0$. This cubic equation has one positive incommensurate root, viz. 1.114, &c. which may be found by its proper rule given in Sect. II. Ch. III. and two impossible. The incommensurate root $y=1.114$, &c. gives $x=19.067$, &c. and these two numbers answer the conditions very nearly.

Quest. VII. The sum of the squares of two numbers (208) and the sum of their cubes (2240) being given, to find them.

Solution.

Notation. Let the sum of the numbers sought be $2x$, and their difference $2y$.

Then the greater number will be $x+y$, and the less $x-y$.

Equation.

Equation.

Therefore $(x+y)^2+(x-y)^2$, or $2x^2+2y^2=208$,
and $y^2=104-x^2$;
$(x+y)^3+(x-y)^3$, or $2x^3+6xy^2=2240$,
and $y^2=\dfrac{1120-x^3}{3x}$.

Resolution. Hence $\dfrac{1120-x^3}{3x}=104-x^2$,
and $x^3-156x+560=0$.

The roots of this equation are 4, 10, and -14.

Answer.

The numbers sought are 12 and 8, because $x=10$, $y=2$, $x+y=12$, and $x-y=8$.

Verification.

$12^2+8^2=144+64=208$, and $12^3+8^3=1728+512$
$=2240$.

Note. The numbers 12 and 8 give the only just solution. If $x=4$, then $y=\sqrt{88}$, and the numbers sought are $4+\sqrt{88}$ and $4-\sqrt{88}$; the last is negative, but they answer the conditions. Lastly, if $x=-14$, then $y=\sqrt{-92}$, an impossible value; but still the two numbers $-14+\sqrt{-92}$, and $-14-\sqrt{-92}$, being inserted, would answer the conditions. But it has been observed, that such solutions are useless, and without meaning.

CHAPTER VII.

OF INDETERMINATE QUESTIONS OF THE FIRST ORDER.

IT was formerly obſerved, (Chap. I.) that if there are more unknown quantities in a queſtion, than equations by which their relations are expreſſed, it is indetermined; or it may admit of an infinite number of anſwers. Other circumſtances, however, may limit the number in a certain manner, and theſe are various, according to the nature of the queſtion. The contrivances by which ſuch queſtions are reſolved, are ſo very different in different caſes, that they cannot be comprehended in general rules. I will now explain only a method, which is very eaſy and uſeful in the ſolution of indeterminate queſtions of the 1ſt order.

Definition. In this ſort of queſtions the *final equation* is that, in which the number of the unknown quantities is the leaſt poſſible.

Queſt. I. Let it be required to find two numbers whoſe ſum is equal to ten times their difference.

Solution.

Putting x and y for the two numbers ſought, it will be $x+y=10\times\overline{x-y}=10x-10y$; hence $x=\frac{11y}{9}$.

Suppoſe

Suppose $y = 1, \quad 2, \quad 3, \ldots, \quad 9$, and so on.
it will be $x = \frac{11}{9}, \ \frac{22}{9}, \ \frac{33}{9}, \ \ldots, \ \frac{99}{9} = 11$, and so on.

But if it be required that x and y shall both be whole numbers, then, because $x = \frac{11y}{9}$, it is, by Division,

$x = y + \frac{2y}{9}$.

Now, suppose $\frac{2y}{9} = A$, you will have $y = \frac{9A}{2}$, or $y = 4A + \frac{A}{2}$.

Again, put $\frac{A}{2} = B$, and therefore $A = 2B$. Hence

$$y = 4A + \frac{A}{2} = 8B + B = 9B,$$

$$x = y + \frac{2y}{9} = 9B + 2B = 11B.$$

							Differences.
If $B =$	1,	2,	3,	4,	&c. *in infinitum.*		+ 1.
then $y =$	9,	18,	27,	36,	&c.		+ 9.
and $x =$	11,	22,	33,	44,	&c.		+11.

Hence the following

RULES. I. *From the final equation draw the value of that unknown quantity (as* x *in our question) which has the less coefficient.*

II. *That value being an improper fraction, find by Division the whole and the fractionary part, which will be a proper fraction.*

III. *Put this fraction equal to a new unknown quantity, as* A, *and from the equation hence arising draw the value of the other unknown quantity (which in our case is* y) *upon which, if fractionary, work as you did upon the value*

value of x, taking a new unknown quantity, as B; *and so on, till you meet with an integer value.*

IV. *Then express the values of* y *and* x *by the quantities contained in that integer value, and they will also be integer.*

V. *Lastly, put your last unknown quantity (as* B *in our example) equal successively to the natural numbers* 0, 1, 2, 3, *&c. or* −1, −2, −3, *&c. and write down in the same column under them the corresponding values of* y *and* x, *among which you will find all the possible answers.*

Notes. I. The values of the quantities sought will always form an arithmetical series, whose difference (that is, the common difference of its terms) may serve to continue it as long as need requires.

II. If the coefficients of the unknown quantities have a common divisor, their values cannot be expressed in whole numbers, unless the known quantity itself be divisible by the same divisor. These coefficients ought to be prime numbers to one another.

Quest. II. A merchant owes 1200*l.* and not being possessed of money, offers two sorts of merchandize in payment. The first is worth 7*l.* an ell, and the second 5*l.* Find in how many different ways, without fractions, he can pay his debt.

Solution.

Take x and y for the numbers of ells of the 1st and 2d merchandize. Then

$$7x+5y=1200, \text{ and } y=\frac{1200-7x}{5}=240-x-\frac{2x}{5}.$$

Sup-

Suppose $\frac{2x}{5}=2A$; hence $x=5A$, and $y=240-7A$.

					Differences.
If $A=$	1,	2,	3,	&c. *in inf.*	$+1$
then $x=$	5,	10,	15,	&c.	$+5$
and $y=$	233,	226,	219,	&c.	-7

Quest. III. A labourer gave lambs in return for sheep. He values each lamb at 4*s.* and each sheep at 9*s.* and he gave 15*s.* into the bargain. I demand in how many different manners his bargain can be changed?

Solution.

Put x for the number of lambs, and y for that of sheep exchanged. Therefore

$4x+15=9y$, and $x=\frac{9y-15}{4}=2y-3+\frac{y-3}{4}$.

Take $y-3=4A$; then $y=4A+3$, and $x=9A+3$.

					Differences.
$A=$	0,	1,	2,	&c. *in inf.*	$+1$
$x=$	3,	12,	21,	&c.	$+9$
$y=$	3,	7,	11,	&c.	$+4$

Quest. IV. I owe a friend a shilling, or some number of shillings, and we have only guineas and louis d'ors about us; the guineas being valued at 21*s.* a piece, and the louis d'ors at 19. How must I acquit myself of this debt?

Solution.

Suppose I give x guineas and receive y louis d'ors.

Then $21x-19y=1$, and $y=\frac{21x-1}{19}=x+\frac{2x-1}{19}$.

Put

Put $2x-1=19A$; whence $x=\frac{19A+1}{2}=9A+\frac{A+1}{2}$.

Again, $A+1=2B$; hence $A=2B-1$, $x=19B-9$, and $y=21B-10$.

					Differences.
$B=$	1,	2,	3,	&c. *in inf.*	$+1$.
$x=$	10,	29,	48,	&c.	$+19$.
$y=$	11,	32,	53,	&c.	$+21$.

Quest. V. To find a ſum of money in pounds and ſhillings, whoſe half is juſt its reverſe.

Note. The reverſe of a ſum of money, as 8*l.* 12*s.* is 12*l.* 8*s.*

Solution.

Let x be the pounds, and y the ſhillings.
The ſum required is $20x+y$ in ſhillings.
Its reverſe is $20y+x$ in ſhillings.

Therefore $\frac{20x+y}{2}=20y+x$ or $20x+y=40y+2x$.

Whence $18x=39y$, or $6x=13y$, and $x=2y+\frac{y}{6}$.

Put $y=6A$; and will be $x=13A$.

					Differences.
$A=$	1,	2,	3,	&c. *in inf.*	$+1$
$x=$	13,	26,	39,	&c.	$+13$
$y=$	6,	12,	18,	&c.	$+6$

From the nature of this queſtion, 13 and 6 are the only two numbers that can give the proper anſwer, viz. 13*l.* 6*s.* for its reverſe 6*l.* 13*s.* is juſt its half.

The ratio of x and y is expreſſed in the loweſt integral terms by 13 and 6; any other expreſſion of it, as the next greater 26 and 12, will not ſatisfy the problem, as 12*l.* 26*s.* is not a proper notation of money in pounds and ſhillings.

Queſt.

Queft. VI. Two country women have together 100 eggs, one fays to the other, when I number my eggs eight by eight I find a remainder of 7; I find the fame remainder, anfwers the other, if I number my own ten by ten. The queftion is, how many eggs has each of them?

Solution.

Becaufe the number of the eggs of the firft country woman, being divided by 8, gives a remainder of 7, and the number of the eggs of the other, being divided by 10, gives the fame remainder, the firft number may be expreffed by $8x+7$ and the fecond by $10y+7$. Therefore,

$$8x+10y+14=100, \text{ and } x=10-y+\frac{3-y}{4}.$$

Make $3-y=4A$, then $y=3-4A$, and $x=5A+7$.

						Diff.
$A=$	1,	0,	-1,	-2,	&c. *in inf.*	-1
$x=$	12,	7,	2,	-3,	&c.	-5
$y=$	-1,	3,	7,	11,	&c.	$+4$
$8x+7=$	103,	63,	23,	-17,	&c.	-40
$10y+7=$	-3,	37,	77,	117,	&c.	$+40$

From this feries it is vifible that there are only the two following anfwers to our queftion, becaufe there are no negative eggs.

1ft. The 1ft country woman has 63 eggs, the 2d 37.
2d. - - - - - - - - - - 23 - - - 77.

Queft. VII. I defire to know how many ladies and gentlemen may be at a ball, if the product of their numbers muft be equal to ten times the excefs of the number of ladies above that of gentlemen.

Solution.

Let x and y be the number of ladies and gentlemen.

Then $xy = 10 \times \overline{x - y}$, and $xy + 10y = 10x$,

or $y = \frac{10x}{x+10} = 10 - \frac{100}{x+10}$.

I take $A = \frac{100}{x+10}$, and I have $x + 10 = \frac{100}{A}$.

Hence $x = \frac{100}{A} - 10$ and $y = 10 - A$.

$$\begin{array}{llllllllll} A = & 1, & 2, & 4, & 5, & 10, & 20, & 25, & 50, & 100. \\ x = & 90, & 40, & 15, & 10, & 0, & -5, & -6, & -8, & -9. \\ y = & 9, & 8, & 6, & 5, & 0, & -10, & -15, & -40, & -90. \end{array}$$

Because the value of the arbitrary quantity A ought to be a divisor of 100 and less than 10, there are only the four following answers.

1st.	The number of the ladies	90, of the gentlem.	9.
2d.	- - - - - - - -	40 - - - - -	8.
3d.	- - - - - - - -	15 - - - - -	6.
4th.	- - - - - - - -	10 - - - - -	5.

Note. This question properly belongs to the 2d order, but I put it here, being easy enough to be resolved on the same principles.

Quest. VIII. Twenty persons consisting of men, women, and children, at a collation paid 20*s.* the men paying 4*s.* a piece, the women 6*d.* a piece, and the children 3*d.* a piece. How many were there of each sort?

Solution.

Let x, y, and z, be the numbers of the men, women, and children. Then reducing shillings into pence, we find

The

$$\begin{array}{ll} \text{The total expence} & 48x+6y+3z=240 \quad \div 3 \\ \hline & 16x+2y+z=80 \\ \text{Number of persons} & x+y+z=20 \\ \hline & 15x+y \quad * = 60 \end{array}$$

Hence $y=60-15x$ and $z=20-y-x=14x-40$.

							Differences.
$x=$	1,	2,	3,	4,	5,	&c.	$+1$
$y=$	45,	30,	15,	0,	-15,	&c.	-15
$z=$	-26,	-12,	2,	16,	30,	&c.	$+14$

Because persons cannot be either negative, or nothing, the only answer is 3 men, 15 women, and 2 children.

Quest. IX. Suppose one would buy 40 birds, consisting of partridges, larks, and quails, for 98*d.* paying 3*d.* a piece for the partridges, a half-penny a piece for the larks, and 4*d.* a piece for the quails. The question is, how many he must have of each sort?

Solution.

Let x, y, and z, be the numbers of the partridges, larks, and quails.

$$\begin{array}{ll} \text{The total expence} & 3x+\frac{y}{2}+4z=98 \quad \times 2 \\ \hline & 6x+y+8z=196 \\ \text{Number of birds} & x+y+z=40 \\ \hline & 5x \quad * +7z=156 \end{array}$$

Hence $x=\frac{156-7z}{5}=31-z+\frac{1-2z}{5}$.

Put $1-2z=5A$; then $z=\frac{1-5A}{2}=-2A+\frac{1-A}{2}$.

Again, $1-A=2B$, and $A=1-2B$.

Therefore $z=-2A+B=5B-2$.

$x=31-z+A=34-7B$.

$y=40-z-x=8+2B$.

									Diff.
$B=$	-1,	0,	1,	2,	3,	4,	5,	&c.	$+1$
$x=$	41,	34,	27,	20,	13,	6,	-1,	&c.	-7
$y=$	6,	8,	10,	12,	14,	16,	18,	&c.	$+2$
$z=$	-7,	-2,	3,	8,	13,	18,	23,	&c.	$+5$

Here you find the following 4 anſwers:

1ſt.	27 partridges,	10 larks,	3 quails.
2d.	20 - - -	12 -	8 - -
3d.	13 - - -	14 -	13 - -
4th.	6 - - -	16 -	18 - -

Queſt. X. There has been a very remarkable comet in a certain year of the Chriſtian æra, in which the number of the ſolar cycle was 9, of the lunar cycle alſo 9, and of the Roman Indiction 3. Which was that year?

Note. The ſolar cycle is a revolution of 28 years.
The lunar 19
The Roman Indiction 15
The Julian period 7980

This period ariſes from the product of the other three revolutions, viz. from $28 \times 19 \times 15$, and it is ſuppoſed to begin 4713 years before our Chriſtian Æra, which therefore has its beginning in the Julian year 4714. Joſeph Julius Scaliger was the author of this account, which is preferable to that of Dionyſius, this being only of 532 years, ariſing from the product of 28 multiplied into 19, and therefore comprehending a leſs number of events in its duration.

Solution

Solution.

Let x be the year of Christ required, $x+4713$ will be the Julian year, which, because it ought to contain the solar cycle a certain number of times (suppose a) and 9 years more, the lunar cycle another certain number of times (suppose b) and also 9 years more, and the Roman Indiction another certain number of times (suppose c), and 3 years more, will give the following equations:

$\frac{x+4713}{28}=a+\frac{9}{28}$ and $x=28a+9-4713$;

$\frac{x+4713}{19}=b+\frac{9}{19}$ and $x=19b+9-4713$;

$\frac{x+4713}{15}=c+\frac{3}{15}$ and $x=15c+3-4713$.

Hence $19b=28a$ or $b=\frac{28a}{19}=a+\frac{9a}{19}$.

Suppose $\frac{9a}{19}=9A$; then $a=19A$,

and $x=28\times 19A+9-4713$.

Therefore $15c=28\times 19A+6=532A+6$;

$$c=\frac{532A+6}{15}=35A+\frac{7A+6}{15}.$$

Put $7A+6=15B$; hence $A=\frac{15B-6}{7}=2B+\frac{B-6}{7}$

Again, $B-6=7C$ or $B=7C+6$. Therefore $A=15C+12$, $x=28\times 19A+9-4713=7980C+1680$. $C=0$ gives $x=1680$ for the year required.

Note. Every other value of C would give such a number of years, that is not yet passed in our Æra, and therefore it cannot belong to the present question.

Quest.

Quest. XI. A grocer has tea at 8*s.* at 10*s.* at 14*s.* per lb. and he proposes to make a mixture of 20lb. of the whole to sell it at 12*s.* per lb. What quantity of each must he take?

Solution,

Let x, y, and z, be the quantities required of the three sorts of tea respectively. Then we shall have the following equations:

$$\frac{x+y+z=20}{4x+4y+4z=80}\times 4$$

$$\frac{8x+10y+14z=240\ (=20\text{lb.}\times 12s.)}{4x+5y+7z=120}\div 2$$

$$\frac{\begin{array}{l}4x+5y+7z=120\\4x+4y+4z=80\end{array}}{*\quad y+3z=40}-$$

Hence $y=40-3z$ and $x=20-y-z=2z-20$.

						Diff.
$z=10,$	11,	12,	13,	14,	&c.	$+1$
$y=10,$	7,	4,	1,	-2,	&c.	-3
$x=0,$	2,	4,	6,	8,	&c.	$+2$

The mixture may be made only in three manners, viz.

1st.	2lb. at 8*s.*	7lb. at 10*s.*	11lb. at 14*s.*
2d.	4lb.	4lb.	12lb.
3d.	6lb.	1lb.	13lb.

Note. Such questions as these are generally resolved by the Rule of Alligation.

Quest.

Queſt. XII. How can I pay 19 livres with half-crowns, ſhillings, and ſix-pences, ſuppoſing 1 livre worth 10*d*.

Solution.

Let x, y, and z, be the number of pieces required reſpectively. Reducing each money into pence, the only equation is

$$\frac{30x + 12y + 6z = 190}{5x + 2y + z = 31\tfrac{4}{6}} \div 6$$

Whence it appears that the queſtion is impoſſible, becauſe it involves fractionary quantities, which cannot be taken away. See Note II. to the Rules after the 1ſt queſtion.

CHAP.

CHAPTER VIII.

A PROMISCUOUS COLLECTION OF QUESTIONS.

I. *Determinate Questions of the 1st Degree.*

1. A Man having a certain number of crowns, said, if $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of them, and 4 more, were added together, then subtracted $\frac{1}{5}$ of them from the sum, they would make 57 crowns. I demand how many he had. Anf. 60.

2. A, B, and C, went together to buy a certain quantity of timber worth 36*l.* B had $\frac{1}{3}$ more than A, and C had $\frac{1}{4}$ more than B. I demand how much each man bought. Anf. A 9*l.* B 12*l.* C 15*l.*

3. A man at his death left 2600*l.* and by his will ordered that his nephew fhould have 60*l.* more than his coufin, and his brother as much as both his nephew and coufin, lefs 100*l.* I demand what fum each is to have. Anf. The coufin 645*l.* the nephew 705*l.* and the brother 1250*l.*

4. A coal merchant being asked how many chaldrons of coals he had fold, anfwered, if the chaldrons were trebled, from the product taking 40 chaldrons, and to the remainder adding $\frac{1}{5}$ of it, I would have fold 4000 chaldron. How many did he fell then? Anf. $1124\frac{4}{9}$.

5. A merchant, to fet up a manufactory, agreed with a perfon who well underftood the bufinefs, that when

when that perſon ſhould attend the manufactory, he would allow him 30*s.* a day: but, that the ſaid perſon was to forfeit 10*s.* for each day he ſhould not attend. At the end of 30 days they ſettled together, and the merchant gave the other 25*l.* How many days did that perſon attend, and how many days did he not? Anſ. 20 days he attended, and 10 days he did not.

6. A man asked how many guineas he had, ſaid, a certain number, multiplied by 8, would produce 7 more than I have; but, if the ſame number be multiplied by 7 only, it will make 5 leſs than I have. What is that number, and how many guineas had he? Anſ. 12 is the number, and 89 are the guineas.

7. One goes with a certain quantity of money about him to a tavern, where he borrows as much as he had then about him, and out of it all ſpends a ſhilling; with the remainder he goes to a ſecond tavern, where he borrows as much as he had left, and there alſo ſpends a ſhilling; and ſo he goes on to a third, and a fourth tavern, borrowing and ſpending as before, after which he had nothing left. I demand how much money he had at firſt about him. Anſ. 11*d.*$\frac{1}{4}$

8. A man and his wife and child dine together at an inn. The landlord charged 10 pence for the child. He charged for the woman as much as for the child and one-third of what he charged for the man; but for the man he charged as much as for the woman and child together. What did he charge for each? Anſ. 30 for the man, and 20 for the woman.

9. Divide 560 into two ſuch parts, that one part may be to the other as 5 to 2. Anſ. 400 and 160.

10. A ſtudent coming into a bookſeller's ſhop, asks the price of his ſtock of books; to which the bookſeller replies, I will have four ſhillings a volume for them one with another; no, replies the ſtudent, I have not money enough by 5*l.* to pay for them at that rate, but if you will let me have them at 3*s.* 4*d.* a volume, I can

then pay for the whole and ſhall have 5*l.* left to pay for the carriage. What number of volumes had the bookſeller, and how much money had the ſtudent? Anſ. 300 volumes, and 55*l.*

11. Given the difference of two numbers 10, and the difference of their ſquares 120, to find the numbers. Anſ. The numbers are 11 and 1.

12. A footman who contracted for 8*l.* a year and a livery, was turned away at the end of 7 months, and received for his wages 2*l.* 13*s.* 4*d.* and his livery. What was the value of the livery? Anſ. 4*l.* 16*s.*

13. A ſhepherd driving a flock of ſheep in time of war, meets with a company of ſoldiers, who plunder him of half his flock and ½ a ſheep over. The ſame treatment he meets with from a ſecond, a third, and a fourth company; every ſucceeding company plundering him of half the flock the laſt had left and ½ a ſheep over, inſomuch that at laſt he had only 8 ſheep left. I demand how many he had at firſt. Anſ. 143.

14. One buys a certain number of eggs, half whereof he buys at 2 a penny, the other half at 3 a penny. Theſe he afterwards ſold out again at the rate of 5 for 2 pence, and, contrary to his expectations, loſt a penny by the bargain. What was the number of eggs? Anſ. 60.

15. It is required to find two numbers with the following properties: that ½ the firſt with $\frac{1}{3}$ part of the ſecond may make 16, and that ¼ of the firſt with $\frac{1}{5}$ of the ſecond may make 9., Anſ. 12 and 30.

16. Divide 20 into two ſuch parts, ſo that $\frac{1}{3}$ of the one part added to $\frac{1}{5}$ of the other may make 6. Anſ. 15 and 5.

17. Says A to B, give me 5*s.* of your money, and I ſhall have twice as much as you will have left. Says B to A, rather give me 5*s.* of your money, and I ſhall have 3 times as much as you will have left. What had each? Anſ. A 11, and B 13.

18. From

18. From the Ladies' Diary, 1708.

When first the marriage knot was tied
Betwixt my wife and me,
My age to her's we found agreed,
As nine doth unto three:
But after ten and half ten years,
We man and wife had been,
Her age came up as near to mine,
As eight is to sixteen.
Now tell me, if you can, I pray,
What was our age o'th' marriage day?

Ans. The man's age was 45, and the wife's 15.

19. A jockey has two horses A and B. He has also two saddles, one valued at 16*l.* the other at 4: now, if he sets the better saddle upon A, and the worse upon B, then A will be worth twice as much as B; but if he sets the better saddle upon B, and the worse upon A, then B will be worth three times as much as A. I demand the values of the horses. Ans. A's value 3*l.* 4*s.* and B's 5*l.* 12*s.*

20. A certain company at a tavern, when they came to pay their reckoning, found, that had there been four more in company they might have paid a shilling a piece less than they did. And that if there had been three fewer in company, they must have paid a shilling a piece more than they did. I demand the number of persons, what each paid, and what was the reckoning. Ans. 24 persons, their quota 7*s.* and their reckoning 8*l.* 8*s.*

21. What two numbers are those, whereof the less is to the greater as 2 is to 5, and the product of whose multiplication is ten times the sum of the numbers? Ans. 14 and 35.

22. To find four numbers with the following properties: the sum of the three first is 13, the sum of

the two first and last is 17, the sum of the first and two last is 18, and the sum of the three last 21. Ans. 2, 5, 6, 10.

II. *Determinate Questions of the 2d and higher Degrees.*

1. Given the product of the multiplication of two numbers $=144$, and the quotient of the greater divided by the less $=9$, to find the two numbers. Ans. 36 and 4.

2. What two numbers are those, whose difference is 15, and half of whose product is equal to the cube of the less? Ans. 3 and 18.

3. There are three numbers in continual geometrical proportion; the sum of the 1st and 2d is 10, and the difference of the 2d and 3d is 24. What are the numbers? Ans. 2, 8, 32.

4. To find two numbers, of which the product is 100, and the difference of their square roots 3. Ans. 4 and 25.

5. To find four continued proportionals, of which the sum of the extremes is 56, and the sum of the means 24. Ans. 54, 18, 6, 2.

6. A certain member of parliament ordered his steward to distribute 7*l.* 4*s.* among the poor voters. When they were assembled together, and the steward ready to distribute the money, there come in unexpectedly two more claimants; by which means, those at first assembled received one shilling a piece less than they otherwise would have done. What was their number at first? Ans. 16 persons.

7. A traveller, A, sets out from a certain place and travels 1 mile the 1st day, 3 miles the 2d day, 5 the 3d day, and so on according to the series of the odd num-

numbers. Eeight days after another, B, ſets out and travels the ſame road at the rate of 36 miles every day. I demand how long and how far A muſt travel before he is overtaken by B? Anſ. 12 days, and 144 miles.

8. A gentleman buys a horſe, which he ſells again for 56 crowns, and gains as many crowns in 100, as the horſe coſt him. Quære the price of the horſe? Anſ. 40 crowns.

9. Two partners, A and B, gain 140*l.* A's money was in trade 3 months, and his gain was 60*l.* leſs than his ſtock; alſo B's money, which was 50*l.* more than A's, was in trade 5 months. Quære A's ſtock? Anſ. 100*l.*

10. Two travellers, A and B, ſet out from two places, C and D, at the ſame time, A from C bound for D, and B from D bound for C; when they met and had computed their travels, it was found that A had travelled 30 miles more than B, and that at their rate of travelling A expected to reach D in 4 days, and B to reach C in 9 days. I demand the diſtance between C and D. Anſ. 150 miles.

11. Two travellers, A and B, ſet out from two places C and D, at the ſame time; A ſets out from C, with a deſign to paſs through D, and B from D, with a deſign to travel the ſame way: after A had overtaken B, and they had computed their travels, it was found, that they had both together travelled 30 miles, that A had paſſed through D 4 days before, and that B at his rate of travelling was a 9 days journey diſtant from C. I demand the diſtance between the two places C and D. Anſ. 6 miles.

12. The governor of a beſieged place, wanting a ſupply of troops, writes to his General, that the garriſon is only as many hundred men, as there are units in the poſitive root of the following equation,

$$x^4 - x^3 - 44x^2 + 49x = 245.$$

The meſſenger is ſtopped by the enemy, and the letter is opened, but nobody is able to declare the meaning of

of it. If you were present, could you have discovered how many hundred men were in the garrison? Anf. 700 men.

13. The difference between the ages of two brothers is 5 years, but the product of the squares of their ages is 4356. What are their ages? Anf. 6 and 11 years.

14. The number of my children, said a father, is such that, if from its 5th power you subtract 50 times its square and 30 times its cube, the remainder will be 4067. How many children had he? Anf. 7 children.

III. *Indeterminate Questions of the 1st Degree.*

1. Suppose one would buy 20 birds for 20*d.* viz. ducks at 2*d.* a piece, partridges at ½*d.* a piece, and geese at 3*d.* a piece. How many must he have of each sort? Anf. 5 ducks, 14 partridges, and a goose.

2. 41 persons, consisting of men, women, and children, at a collation paid 40*s.* the men paying 4*s.* a piece, the women 3*s.* a piece, and the children 4*d.* a piece. How many were there of each sort? Anf. 5 m. 3 w. and 33 ch.

3. Let it be proposed to find in what year of the Christian Æra there has been 17 of the solar cycle, 6 of the lunar cycle, and 5 of the Roman indiction. Anf. in the year 1772.

PART

PART III.

THE ALGEBRAICAL LANGUAGE,

CHAPTER I.

THE NATURE OF THE ALGEBRAICAL LANGUAGE.

INvestigations by common Arithmetic are greatly limited, from the want of characters to express the quantities that are unknown, and their different relations to one another, and to such as are known. Hence letters and other convenient symbols have been introduced to supply this defect; and thus gradually has arisen the science

ſcience of *Algebra*, properly called *Univerſal Arithmetic*, and ſometimes, for its ſymbols, *ſpecious* or *literal*.

In the preceding mathematical languages, the given numbers diſappear in the courſe of the operation, ſo that general rules can ſeldom be derived from it; but in Algebra the known quantities as well as the unknown may be expreſſed by letters, which, through the whole operation, retain their original form; and hence may be deduced, not only general canons for like caſes, but the dependence of ~~the ſeveral~~ quantities concerned, and likewiſe the determination of a problem, without exhibiting which, it is not completely reſolved. This general manner of expreſſing quantities alſo, and the general reaſonings concerning their connections, which may be founded on it, have rendered this ſcience not leſs uſeful in the demonſtration of theorems, than in the reſolution of problems.

To the algebraic alphabet belong not only the letters, but the numbers themſelves, as *numerical* coefficients of quantities expreſſed by letters. Again, theſe letters are not reſtrained to repreſent any particular quantity, as do numbers and lines, but every ſpecies of quantity as well as quantity in general. In an arithmetical problem then, theſe letters repreſent certain numbers (as principal and intereſt, &c.) In geometrical problems they repreſent certain lines (as length and breadth); or ſurfaces; or ſolids. In ſtatical problems they repreſent certain weights; in mechanics certain forces, &c. Theſe letters then are repreſentatives of quantity in general, or in the abſtract; without reſtraining them to ſignify either any particular degree of quantity, or even any particular ſpecies of quantity. Thus, if 10 pounds be the principal, then at 5 per cent. I can find, from the Rule of Three, that 10*s*. is the intereſt. But, if I put a for the principal, be it what it will, and b for the intereſt; if I am told that $a=20b$, I get a general rule for the intereſt in all caſes at 5 per cent.

In

In like manner *a* may ſtand for the length and *b* for the breadth of certain figures, and thus general rules may be eſtabliſhed, ſhewing, in every one of theſe figures, the relation of the length to the breadth. Algebraic rules, in which the ſymbols ſtand for ſuch very general ideas, will of courſe be applicable to every ſubject into which mathematical reaſoning can be introduced.

It is from this univerſal application, that Algebra has been called *the ſcience of quantity in general*; and this ſcience is what I call here *The Algebraical Language*, the nature whereof may be comprehended in the following.

General Principle.

The *known* quantities of a problem, as well as the *unknown*, muſt be expreſſed by *letters*, and their *relations* contained in it, or the *conditions* of it, as they are called, by *equations*. Theſe equations being reſolved by their proper rules, will give a *general ſolution*, from which not only a *general rule* or *theorem* may be drawn, for ſolving all particular caſes of problems of the like kind, but alſo the *neceſſary limitations* of the *data*, for a perfect ſolution of the problem, as well as a *ſynthetical demonſtration* of the ſolution itſelf.

Note. The word *problem* is more general than that of *queſtion*, which is uſed more properly in Arithmetic: but the diviſion of problems is the ſame with that of queſtions. (See Part II. Chap. I.)

CHAPTER II.

OF WRITING ALGEBRAICALLY.

WE *write algebraically* when by letters and mathematical ſigns we repreſent the known and unknown quantities in the problem propoſed, as well as the operations required in it; and alſo, when by general equations we expreſs the conditions of the problem, that is the relations of the known and unknown quantities, contained in it.

Hence the operations upon letters, and the expreſſion of problems become the two chief diviſions of this Chapter.

The expreſſion of problems is the ſame as the expreſſion of queſtions, only that the known quantities of queſtions are expreſſed by numbers, and thoſe of problems by the firſt letters of the common alphabet, as *a*, *b*, *c*, &c. (See Part II. Chap. II.)

The fundamental operations upon letters are performed as thoſe upon letters and numbers. (Ibidem.) But there are ſome other operations, which ought to be explained in the following Sections.

SECTION I.

OF FRACTIONS.

THE operations upon algebraic fractions follow the ſame rules as the operations upon common fractions. (See Part I. Chap. V.) It is only to be remarked, that when the greateſt common diviſor of algebraical quantities is required, any common *ſimple* diviſor is eaſily found by inſpection. But, when the greateſt *compound* diviſor is wanted, the common rule (p. 80.) is to be applied, with the following remarks.

The ſimple diviſors of each of the quantities are to be taken out; the remainders in the ſeveral operations are alſo to be divided by their ſimple diviſors; and the quantities are always to be ranged according to the powers of the ſame letter.

The ſimple diviſors in the given quantities, or in the remainders, do not affect a compound diviſor which is wanted; and hence alſo to make the diviſion ſucceed, any of the dividends may be multiplied by a ſimple quantity. Beſides the ſimple diviſors in the remainders not being found in the diviſors from which they ariſe, can make no part of the common meaſure or diviſor ſought; and for the ſame reaſon, if in ſuch a remainder, there be any compound diviſor which does not meaſure the diviſor from which it proceeds, it may be taken out.

Examples.

$$\text{I.}\quad a^2-b^2\ )\ a^2-2ab+b^2\ (\ 1$$
$$a^2-\ b^2$$
$$\text{Rem.}\quad *\ -2ab+2b^2$$
$$\div -2b$$
$$a-b\ )\ a^2-b^2\ (\ a+b$$
$$a^2-b^2$$
$$*\quad *$$

Note. In the following ſecond Example, both the *given quantities* are divided by their *ſimple common diviſor b*; then each of the *quotients* hence ariſing is divided by its *particular ſimple diviſor*; and, becauſe the firſt term $3a^3$ of the dividend is not diviſible by the 1ſt term $4a^2$ of its correſpondent diviſor, that dividend is then multiplied by 4. Again, the remainder is divided by its *ſimple diviſor b*, and the quotient hence ariſing is multiplied by 4, to have $12a^2$ diviſible by $4a^2$. Laſtly, the ſimple diviſor $19b$ of the remainder is taken out; and then the operation is brought to its end by the common rule.

II.

II.

$$
\begin{array}{lll}
8a^2b^2 - 10ab^3 + 2b^4 \;) & 9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4 & \div b \\
\hline
8a^2b - 10ab^3 + 2b^3 \;) & 9a^4 - 9a^3b + 3a^2b^2 - 3ab^3 & \\
\quad\quad\quad\quad 2b & \quad\quad\quad\quad\quad\quad 3a & \\
\hline
4a^2 - 5ab + b^2 \;\div\;) & 3a^3 - 3a^2b + ab^2 - b^3 & \div \\
\hline
 & & \times 4 \\
4a^2 - 5ab + b^2 \;) & 12a^3 - 12a^2b + 4ab^2 - 4b^3 \;(& 3a+3 \\
 & 12a^3 - 15a^2b + 3ab^2 & \\
\hline
 & * \quad 3a^2b + ab^2 - 4b^3 & \div b \\
\hline
 & 3a^2 + ab - 4b^2 & \times 4 \\
\hline
 & 12a^2 + 4ab - 16b^2 & \\
 & 12a^2 - 15ab + 3b^2 & \\
\hline
 & * \quad 19ab - 19b^2 & \div 19b \\
\hline
 & a - b &
\end{array}
$$

$a-b$

$$\begin{array}{r|l}
a-b\ ) & 4a^2-5ab+b^2\ (\ 4a-b \\
& 4a^2-4ab \\
\hline
& *\ -\ ab+b^2 \\
& \quad -\ ab+b^2 \\
\hline
& \quad\ *\quad *
\end{array}$$

In the firſt example, $a-b$ is the greateſt common diviſor required; but in the ſecond that diviſor is $\overline{a-b}\times b=ab-bb$, for there is a ſimple common diviſor b, and a compound $a-b$.

SECTION II.

OF INVOLUTION AND EVOLUTION.

LEMMA. The reciprocals of the powers of a quantity may be expreſſed by that quantity with negative exponents of the ſame denomination. That is, the ſeries

$$a,\ 1,\ \frac{1}{a},\quad \frac{1}{a^2},\quad \frac{1}{a^3},\ldots,\frac{1}{a^m},\ \&c.$$

may be expreſſed by

$$a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ldots,a^{-m},\ \&c.$$

For the rule for dividing the powers of the ſame root was to ſubtract the exponents. (See Part II. Ch. II. Note V. upon the fundamental Operations.) If then the index of the diviſor be greater than that of the dividend, the index of the quotient muſt be negative.

Thus,

Thus, $\frac{a^2}{a^3}=a^{2-3}=a^{-1}$. Also $\frac{a^2}{a^3}=\frac{1}{a^{-1}}$

$$\frac{a^m}{a^m}=a^{m-m}=a^0. \quad \text{And } \frac{a^m}{a^m}=1.$$

Cor. I. Hence any quantity which multiplies either the numerator or denominator of a fraction, may be transposed from the one to the other, by changing the sign of its index.

Thus, $\frac{x}{y^2}=xy^{-2}$. And $\frac{a^2x}{y^3}=\frac{a^2}{y^3x^{-1}}$.

Cor. II. From this notation, it is evident that these *negative powers*, as they are called, are multiplied by adding, and divided by subtracting their exponents.

Thus, $a^{-2}\times a^{-3}=a^{-5}$. Or $\frac{1}{a^2}\times\frac{1}{a^3}=\frac{1}{a^5}=a^{-5}$

$$a^{-2}\div a^{-3}=a^1. \quad \text{Or } \frac{1}{a^2}\div\frac{1}{a^3}=\frac{a^3}{a^2}=a^1.$$

I. Of Involution.

PROPOSITION. *To involve any algebraic quantity.*

Case I.

When the quantity is simple.

RULE. *Multiply the exponents of the letters by the index of the power required, and raise the numeral coefficient to the same power.*

Thus, the 2d power of a is $a^{1\times 2}=a^2$.

The

The 3d power of $2a^2$ is $8a^{2\times 3}=8a^6$.
The 3d power of $3ab^3$ is $27a^{1\times 3}b^{3\times 3}=27a^3b^9$.

For, the multiplication would be performed by the continued addition of the exponents, and this multiplication of them is equivalent.

Thus, $a^1\times a^1=a^{1+1}=a^2=^{1\times 2}$.
$8a^2\times a^2\times a^2=8a^6=8a^{2\times 3}$.
$27a^1\times a^1\times a^1\times b^3\times b^3\times b^3=27a^3b^9$.

Note. The ſame rule holds alſo, when the ſigns of the exponents are negative.

Rule for the Signs.

If the ſign of the given quantity is +, all its powers muſt be poſitive. If the ſign is −, then all its powers whoſe exponents are even numbers, are poſitive; and all its powers whoſe exponents are odd numbers, are negative.

This is obvious from the Rule for the Signs in Multiplication (p. 74.)

Caſe II.

When the quantity is compound.

RULE. *The powers muſt be found by a continual multiplication of that quantity by itſelf.*

Thus, the ſquare of $x+\frac{a}{2}$ is found by multiplying it into itſelf. The product is $x^2+ax+\frac{a^2}{4}$. The cube of $x+\frac{a}{2}$ is got by multiplying the ſquare already found

by

by the root, which multiplied into the cube gives the biquadrate or 4th power, and ſo on. (See Sect. IX. p. 84.)

Notes. I. The ſquare of a binomial conſiſts of the ſquares of the two parts, and twice the product of the two parts.

II. Fractions are raiſed to any power by raiſing both numerator and denominator to that power, as is evident from the rule for multiplying fractions. (See p. 42, and 95.)

III. The involution of compound quantities is rendered much eaſier by the binomial theorem; for which ſee the following Chap. IV. Sect. II. where of Infinite Series.

II. Of Evolution.

PROPOSITION. *To evolve any algebraic quantity.*

General Rule for the Signs.

1. *The root of any poſitive power may be either poſitive or negative, if it is denominated by an even number; if the root is denominated by an odd number, it is poſitive only.*

2. *If the power is negative, the root alſo is negative, when it is denominated by an odd number.*

3. *If the power is negative, and the denomination of the root even, then no root can be aſſigned.*

Notes. I. This rule is eaſily deduced from that given in Involution.

II. The root of a positive power, denominated by an even number, has often the sign $\pm$ before it, denoting that it may have either $+$ or $-$.

III. The root of a negative power, denominated by even number, is expressed by setting the radical sign before that power, and then it is called an *impossible* or *imaginary* root, as $\pm\sqrt{-4}$, $\pm\sqrt[4]{-3}$.

IV. The radical sign may be employed to express any root of any quantity whatever, as is known; but sometimes the root may be accurately found by the following Rules, and when it cannot, it may often be more conveniently expressed by a series.

Case I.

When the quantity is simple.

RULE. *Divide the exponents of the letters by the index of the root required, and prefix the root of the numeral coefficient.*

1. The exponents of the letters may be multiples of the index of the root, and the root of the coefficient may be extracted.

Thus, the square root of a^4 is $a^{4\div 2}=\pm a^2$.

$$\sqrt[3]{27a^6}=3a^{6\div 3}=3a^2.$$

$$\sqrt[4]{a^4b^{12}}=a^{\frac{4}{4}}b^{\frac{12}{4}}=\pm a^1b^3.$$

2. The exponents of the letters may not be multiples of the index of the root, and then they become fractions; and when the root of the numerical coefficient cannot be extracted, it may also be expressed by a fractional exponent, its original index being understood to be 1.

Thus,

Thus, $\sqrt{16a^3b^2} = \pm 4ab\sqrt{a} = \pm 4a^{\frac{3}{2}}b$.

$\sqrt[3]{7ax^3} = x\sqrt[3]{7a} = a^{\frac{1}{3}}x\sqrt[3]{7} = 7^{\frac{1}{3}}a^{\frac{1}{3}}x$.

Notes. I. As Evolution is the reverfe of Involution, the reafon of the Rule is evident.

II. The root of any fraction is found by extracting that root out of both numerator and denominator. (See p. 96.)

Cafe II.

When the quantity is compound.

1. *To extract the Square Root.*

RULES. I. *The given quantity is to be ranged according to the powers of the letters, as in Divifion.*

II. *The fquare root is to be extracted out of the 1ft term (by the preceding Rules) which gives the firft part of the root fought. Subtract its fquare from the given quantity, and divide the firft term of the remainder by double the part already found, and the quotient is the fecond term of the root.*

III. *Add this fecond part to double of the firft, and multiply their fum by the fecond part: fubtract the product from the laft remainder, and if nothing remain, the fquare root is obtained. But if there is a remainder, it muft be divided by the double of the roots already found, and the quotient will give the third part of the root; and fo on.*

Example I.

$$\sqrt{(a^2+2ab+b^2)}=a+b \quad \text{by Rule II.}$$

$$-a^2$$

$$\text{Rule II. } 2a)\ * \quad 2ab+b^2 \quad \frac{2a+b}{} \times b \quad \text{by Rule III.}$$

$$2ab+b^2=2ab+b^2$$

$$* \quad *$$

Example II.

$$\sqrt{(x^4-ax^2+\frac{a^2}{4}(=x^2-\frac{a}{2}} \quad \text{by Rule II.}$$

$$x^4$$

$$\text{Rule II.} \quad \overline{} \qquad 2x^2-\frac{a}{2}$$

$$2x^2)*\ -ax^2+\frac{a^2}{4} \quad \overline{} \times -\frac{a}{2} \quad \text{by Rule III.}$$

$$-ax^2+\frac{a^2}{4}=-ax^2+\frac{a^2}{4}$$

$$* \quad *$$

Note. The reafon of thefe Rules appears from the compofition of a fquare. (See Note I. Cafe II. of Involution.)

2. To

2. *To extract any other Root.*

RULE. *Range the quantity according to the dimensions of its letters, and extract the said root out of the first term, and that shall be the first part of the root required. Then raise this root to a dimension lower by unity, than the number that denominates the root required, and multiply the power that arises, by that number itself: divide the second term of the given quantity by the product, and the quotient shall give the second part of the root required.* In like manner are the other parts to be found, by considering those already got as making one term.

Example.

$$\sqrt{}(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5)=a+b$$

$$a^5$$

$$\overline{}$$

$$5a^4)\ *\ \ 5a^4b(b$$

And $a+b$ raised to the 5th power is the given quantity, and therefore it is the root sought.

Notes. I. In Evolution it will often happen, that the operation will not terminate, and the root will be expressed by a series, as in the following

Example,

Example.

$$\sqrt{(a^2+x^2)}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}, \ \&c.$$

$$\begin{array}{lll}
 & a^2 & \\
\hline
2a) & * & x^2 \\
 & & x^2+\dfrac{x^4}{4a^2} \\
\hline
2a) & * & -\dfrac{x^4}{4a^2} \\
 & & -\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6} \\
\hline
2a) & * & \dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6} \\
 & & \dfrac{x^6}{8a^4}+\dfrac{x^8}{16a^6}-\dfrac{x^{10}}{64a^8}+\dfrac{x^{12}}{256a^{10}} \\
\hline
 & & * -\dfrac{5x^8}{64a^6}+\dfrac{x^{10}}{64a^8}-\dfrac{x^{12}}{256a^{10}} \\
 & & \quad \&c. \qquad \&c. \qquad \&c.
\end{array}$$

Notes. I. The Extraction of Roots by series is much facilitated by the binomial theorem. (See the following Chap. IV. Sect. II. where of Infinite Series.)

II. By similar Rules, founded on the same principles, are the roots of numbers to be extracted.

SEC-

SECTION III.

OF SURDS.

DEFINITION. Quantities with fractional exponents are called *Surds* or *Imperfect Powers.*

Notes. I. Such quantities are alſo called *irrational*, in oppoſition to others with integral exponents, which are called *rational.* (See the Note p. 95.)

II. Surds may be expreſſed either by the fractional exponents, or by the radical ſign, the denominator of the fraction being its index; and hence the orders of ſurds are denominated from this index. In the following operation, however, it is generally convenient to uſe the notation by the fractional exponents. Thus,

$$\sqrt[3]{a}=a^{\frac{1}{3}};\ \sqrt{4ab^2}=2a^{\frac{1}{2}}b;\ \sqrt[4]{a^3b^2}=a^{\frac{3}{4}}b^{\frac{2}{4}}.$$

III. A rational quantity may be put in the form of a ſurd, by reducing its index to the form of a fraction of the ſame value. Thus,

$$a=a^1=a^{\frac{2}{2}}=a^{\frac{3}{3}}=a^{\frac{m}{m}}=\sqrt[m]{a^m}$$

$$a^2b=a^{\frac{2\times m}{m}}b^{\frac{1\times m}{m}}=a^{\frac{2m}{m}}b^{\frac{m}{m}}=\sqrt[m]{a^{2m}b^m}.$$

Principle:

Principle. *If the numerator and denominator of a fractional exponent be both multiplied, or both divided by the same quantity, the value of the power is the same.* Thus,

$$a^{\frac{2}{6}} = a^{\frac{2 \times 2}{2 \times 6}} = a^{\frac{4}{12}} = a^{\frac{4 \times 5}{12 \times 5}} = a^{\frac{20}{60}} = \text{\&c.}$$

$$a^{\frac{m}{n}} = a^{\frac{m \times c}{n \times c}} = a^{\frac{cm}{cn}} = a^{\frac{cm \times d}{cn \times d}} = a^{\frac{cdm}{cdn}} = \text{\&c.}$$

This principle is manifeſt from the nature of fractions, (ſee Principle IX. p. 24.) and upon it depend the operations concerning Surds.

LEMMA. A rational quantity may be put into the form of a ſurd, by reducing its exponent to the form of a fraction of the ſame value.

Thus, $a^1 = a^{\frac{2}{2}} = \sqrt{a^2}$; $a^2c = a^{\frac{6}{3}}c^{\frac{3}{3}} = \sqrt[3]{a^6c^3}$.

PROPOSITION I. To reduce Surds of different denominations to others of the ſame value, and of the ſame denomination.

RULE. *Write down the given ſurds with their fractional exponents, if they already are not; then reduce theſe exponents to others of the ſame value, and having the ſame common denominator.*

Examples.

Examples.

I. Reduce $\sqrt{a}$ and $\sqrt[3]{b^2}$ to the same index or denomination.

$$\sqrt{a}=a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}$$

$$\sqrt[3]{b^2}=b^{\frac{2}{3}}=b^{\frac{4}{6}}=\sqrt[6]{b^4}.$$

II. Reduce $\sqrt{ab}$, $\sqrt[3]{c^2g}$, and $\sqrt[5]{e^3g^2}$ to the same index.

$$\sqrt{ab}=a^{\frac{1}{2}}b^{\frac{1}{2}}=a^{\frac{15}{30}}b^{\frac{15}{30}}=\sqrt[30]{a^{15}b^{15}}$$

$$\sqrt[3]{c^2g}=c^{\frac{2}{3}}g^{\frac{1}{3}}=c^{\frac{20}{30}}g^{\frac{10}{30}}=\sqrt[30]{c^{20}g^{10}}$$

$$\sqrt[5]{e^3g^2}=e^{\frac{3}{5}}g^{\frac{2}{5}}=e^{\frac{18}{30}}g^{\frac{12}{30}}=\sqrt[30]{e^{18}g^{12}}.$$

III. Reduce $\sqrt[m]{a^n}$, $\sqrt[c]{p^e}$, and $\sqrt[r]{q^u}$, to the same index.

$$\sqrt[m]{a^n}=a^{\frac{n}{m}}=a^{\frac{cnr}{cmr}}=\sqrt[cmr]{a^{cnr}}$$

$$\sqrt[c]{p^e}=p^{\frac{e}{c}}=p^{\frac{emr}{cmr}}=\sqrt[cmr]{p^{emr}}$$

$$\sqrt[r]{q^u}=q^{\frac{u}{r}}=q^{\frac{cmu}{cmr}}=\sqrt[cmr]{q^{cmu}}$$

Note. If ſurds have coefficients, theſe may be worked by the ſame Rule, or even remain untouched. (See the Lemma.)

Example.

Reduce $a\sqrt[m]{c^n}$ and $4\sqrt[e]{p^u}$ to the ſame index.

$$a\sqrt[m]{c^n} = a^{\frac{1}{1}}\,c^{\frac{n}{m}} = a^{\frac{em}{em}}\,c^{\frac{en}{em}} = \sqrt[em]{a^{em}c^{en}} = a\sqrt[em]{c^{en}}$$

$$4\sqrt[e]{p^u} = 4^{\frac{1}{1}}\,p^{\frac{u}{e}} = 4^{\frac{em}{em}}\,p^{\frac{mu}{em}} = \sqrt[em]{4^{em}p^{mu}} = 4\sqrt[em]{p^{mu}}.$$

PROP. II. To multiply and divide Surds.

RULES. I. *When they are ſurds of the ſame rational quantity, add and ſubtaaƈt their exponents.*

Thus, $\sqrt[3]{a} \times \sqrt[4]{a^3} = a^{\frac{1}{3}} \times a^{\frac{3}{4}} = a^{\frac{1}{3}+\frac{3}{4}} = a^{\frac{13}{12}} = \sqrt[12]{a^{13}}$

$$\frac{\sqrt{(a^2-b^2)}}{\sqrt[3]{(a^2-b^2)}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{(a^2-b^2)^{\frac{1}{3}}} = (a^2-b^2)^{\frac{1}{2}-\frac{1}{3}} = (a^2-b^2)^{\frac{1}{6}}$$

$$= \sqrt[6]{(a^2-b^2.)}$$

II. *If they are ſurds of different rational quantities, let them be brought to others of the ſame denomination, if already they are not, by* Prop. I. *Then, by multiplying or dividing theſe rational quantities, their produƈt or quotient may be ſet under the common radical ſign.*

Thus,

Thus, $\sqrt[m]{a}\times\sqrt[n]{c}=a^{\frac{1}{m}}c^{\frac{1}{n}}=a^{\frac{n}{mn}}c^{\frac{m}{mn}}=\sqrt[mn]{a^n c^m}$.

$$\frac{\sqrt{(a^2-b^2)}}{\sqrt{(a+b)}}=\sqrt{\frac{a^2-b^2}{a+b}}=\sqrt{(a-b)}.$$

$$\frac{\sqrt[4]{a^3c^2}}{\sqrt[3]{a^2c}}=\frac{a^{\frac{3}{4}}c^{\frac{2}{4}}}{a^{\frac{2}{3}}c^{\frac{1}{3}}}=a^{\frac{3}{4}-\frac{2}{3}}c^{\frac{2}{4}-\frac{1}{3}}=a^{\frac{1}{12}}c^{\frac{2}{12}}=\sqrt[12]{ac^2}.$$

$$\frac{\sqrt[m]{a}}{\sqrt[n]{c}}=\frac{a^{\frac{1}{m}}}{c^{\frac{1}{n}}}=\frac{a^{\frac{n}{mn}}}{c^{\frac{m}{mn}}}=\sqrt[mn]{\frac{a^n}{c^m}}.$$

Notes. I. If the furds have any rational coefficients, their product or qaotient muft be prefixed.

Thus, $m\sqrt{a}=n\sqrt{c}=mn\sqrt{ac}$

$$m\sqrt{a}\div n\sqrt{c}=\frac{m}{n}\sqrt{\frac{a}{c}}.$$

II. It is often convenient in thefe operations, not to bring the furds of fimple quantities to the fame index, but to exprefs their product or quotient without the radical fign, in the fame manner as if they were rational quantities.

Thus, $\sqrt[m]{a}\times\sqrt[n]{c}=a^{\frac{1}{m}}c^{\frac{1}{n}}$ and $\frac{\sqrt[4]{a^3c^2}}{\sqrt[3]{a^2c}}=a^{\frac{1}{12}}c^{\frac{1}{6}}$.

Corollary. If a rational coefficient be prefixed to a radical ſign, it may be reduced to the form of a ſurd by the Lemma, and multiplied by this Propoſition; and converſely, if the quantity under the radical ſign be diviſible by a perfect power of the ſame denomination, it may be taken out, and its root prefixed as a coefficient.

Thus, $a\sqrt{c}=\sqrt{a^2c}$; $2\sqrt[3]{a}=\sqrt[3]{8a}$; $2a\sqrt{(1-2c)}$ $=\sqrt{(4a^2-8a^2c)}$.

Conv. $\sqrt{a^2c}=a\sqrt{c}$; $\sqrt[3]{8a}=2\sqrt[3]{a}$; $\sqrt{(4a^2-8a^2c)}$ $=2a\sqrt{(1-2c)}$.

Note. Even when the quantity under the radical ſign is not diviſible by a perfect power, it may be uſeful ſometimes to divide ſurds into their component factors, by reverſing the operation of this Propoſition.

Thus, $\sqrt{(a^2-x^2)}=\sqrt{(a+x)}\times\sqrt{(a-x)}$

$\sqrt[3]{(a^2c-cx^2)}=\sqrt[3]{(a+x)}\times\sqrt[3]{(a-x)}\times\sqrt[3]{c}$.

PROP. III. To involve or evolve ſurds.

This is performed by the ſame Rules as in other quantities, by multiplying or dividing their exponents by the index of the power or root required.

Thus, the 3d power of $\sqrt{a}$ is $a^{\frac{1}{2}\times 3}=a^{\frac{3}{2}}=\sqrt{a^3}$.

the 3d root of $\sqrt{a}$ is $a^{\frac{1}{2}\div 3}=a^{\frac{1}{6}}=\sqrt[6]{a}$.

In

In general $\left(\sqrt[n]{a^c}\right)^m = a^{\frac{c}{n}\times m} = a^{\frac{cm}{n}} = \sqrt[n]{a^{cm}}$

$$\sqrt[m]{\sqrt[n]{a^c}} = a^{\frac{c}{n}\div m} = a^{\frac{c}{mn}} = \sqrt[mn]{a^c}.$$

Cor. Hence it appears, that $\sqrt{\sqrt{a}} = \sqrt[4]{a}$, $\sqrt[3]{\sqrt{a}} = \sqrt[6]{a}$, $\sqrt{\sqrt[3]{a}} = \sqrt[6]{a}$, and in general $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Note. The notation by negative exponents, mentioned in the Lemma at the beginning of Sect. II. of this Chapter, is applicable to fractional exponents, in the same manner as to integers.

$$\text{Thus, } \frac{a}{\sqrt[n]{c^m}} = \frac{a}{c^{m\div n}} = ac^{-m\div n}.$$

SECTION IV.

OF LOGARITHMS.

I. *Of Logarithms in General.*

LET there be a series of powers expressed by the help of their indexes, the common root or fundamental quantity being always the same, as

$$\ldots\ldots, a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, \ldots\ldots$$

DE-

DEFINITION. The indexes of these powers are called the *logarithms* of the same powers. Thus, 3 is the logarithm of a^3, -3 is the logarithm of a^{-3}, and in general m is the logarithm of a^m.

Note. The logarithms are marked by the letter L or l. Thus, when you write down $m = \text{L}.a^m$, or $m = \text{l}.a$, you mean, that m is equal to the logarithm of the power or quantity a^m. Some authors take the mark log. Thus $m = \log.a^m$ means that m is the logarithm of a^m.

Cor. I. Because $a^0 = 1$, $a^{-1} = 1 \div a$, $a^{-2} = 1 \div a^2$, and generaly $a^{-m} = 1 \div a^m$, by the Lemma of the second Section of this Chapter, we shall have by the preceding Definition and Note, $0 = \text{l}.a^0 = \text{l}.1$, $-1 = \text{l}.a^{-1} = \text{l}.1 \div a$, $-2 = \text{l}.a^{-2} = \text{l}.1 \div a^2$, and generally $-m = \text{l}.a^{-m} = \text{l}.1 \div a^m$. That is, whatever be the integer value of the root a, the logarithm of unity is 0, and the logarithm of a fraction, whose numerator is 1, and denominator some perfect power of a, is expressed by a negative number.

Cor. II. Suppose now, $m = \frac{1}{2}$; this value of m substituted in the equation $m = \text{l}.a^m$, will give $\frac{1}{2} = \text{l}.a^{\frac{1}{2}}$, or $\frac{1}{2} = \text{l}.\sqrt{a}$; if $m = \frac{1}{3}$, it will be $\frac{1}{3} = \text{l}.a^{\frac{1}{3}} = \text{l}.\sqrt[3]{a}$; and in general, if $m = \frac{1}{n}$, n being a whole number, we shall find $\frac{1}{n} = \text{l}a^{\frac{1}{n}} = \text{l}.\sqrt[n]{a}$. That is, the logarithm of an imperfect power of the root a is represented by the fractional index of that power.

Cor.

Cor. III. Becauſe the root a is underſtood to be always the ſame, and therefore always poſitive, all its powers will alſo be poſitive. Hence no logarithms can be aſſigned for negative quantities, or the logarithms of negative quantities are impoſſible, it being impoſſible to get a negative power from a poſitive root.

Cor. IV. Take $a^m = b$, then $\mathrm{l}.a^m = \mathrm{l}.b$, but $m = \mathrm{l}.a^m$; therefore $m = \mathrm{l}.b$, which value of m changes the equation $a^m = b$ in this other $a^{\mathrm{l}.b} = b$. That is, any quantity whatever is ſuch a power of our root a, as is expreſſed by an index repreſenting the logarithm of the ſame quantity. Thus, $c = a^{\mathrm{l}.c}$, $d = a^{\mathrm{l}.d}$, &c.

Cor. V. Being $b = a^{\mathrm{l}.b}$, and $c = a^{\mathrm{l}.c}$; the product of theſe equations will be $bc = a^{\mathrm{l}.b} \times a^{\mathrm{l}.c}$. But $a^{\mathrm{l}.b} \times a^{\mathrm{l}.c} = a^{\mathrm{l}.b + \mathrm{l}.c}$. (See Part II. Chap. II. Note IV. upon the Fundamental Operations.) Therefore $bc = a^{\mathrm{l}.b + \mathrm{l}.c}$. Now the index is always the logarithm of the power, by Definition; and this power in our caſe is expreſſed by bc. Therefore $\mathrm{l}.bc = \mathrm{l}.b + \mathrm{l}.c$. In the ſame manner we get $\mathrm{l}.bcd = \mathrm{l}.b + \mathrm{l}.c. + \mathrm{l}.d$. That is, the logarithm of a product is found by adding together the logarithms of its factors.

Cor. VI. If inſtead of multiplying, we divide the 1ſt equation by the 2d, we ſhall have $\frac{b}{c} = \frac{a^{\mathrm{l}.b}}{a^{\mathrm{l}.c}}$, or $\frac{b}{c} = a^{\mathrm{l}.b - \mathrm{l}.c}$. (See Part II. Chap. II. Note V. upon the

the Fundamental Operations.) Hence $l.\frac{b}{c} = l.b - l.c$ by Definition. That is, the logarithm of a fraction is found by ſubtracting the logarithm of the denominator from that of the numerator.

Cor. VII. Whence it follows, that the logarithm of a proper fraction will always be a negative number, becauſe in this ſuppoſition a greater number is always to be ſubtracted from a leſs. Thus, for inſtance, $l.\frac{2}{3} = l.2 - l.3$, which difference muſt be a negative number, for the logarithm of 3 ought to be greater than that of 2.

Cor. VIII. It appears from Cor. V. and VI. that the addition and ſubtraction of logarithms anſwers to the multiplication and diviſion of the natural numbers to which they belong. Therefore, whenever one number is to be multiplied or divided by another, it is but adding their logarithms together in the 1ſt caſe, or ſubtracting the logarithm of the diviſor from the logarithm of the dividend in the 2d: becauſe that ſum, and this difference will give a third logarithm, whoſe natural number will be the product, or the quotient required.

Cor. IX. Suppoſe $b = c$, then the equation $l.bc = l.b + l.c$ will become $l.cc = l.c + l.c$, or $l.c^2 = 2l.c$. Again, if $b = c = d$, the equation $l.bcd = l.b + l.c + l.d$ will become $l.ccc = l.c + l.c + l.c$, or $l.c^3 = 3l.c$. Hence in general $l.c^n = nl.c$. That is, the logarithm of a power will be found by taking the logarithm of its root ſo many times as there are units in the index.

Cor. X. Becauſe $l.c^n = nl.c$, we get $l.c = \frac{l.c^n}{n}$. That is, the logarithm of a root will be found by dividing the logarithm of the given power by its index.

Cor. XI. Hence it follows, that the raiſing a number to any given power, or the extracting any root out of

of it, is easily performed, only by multiplying the logarithm of that number by the index of the given power, or dividing it by the index of the given root.

Note. From these operations upon logarithms will easily appear the very great usefulness of a good table of logarithms, in order to facilitate the multiplication, division, involution and evolution of natural numbers, which cannot but be very useful, not only in Trigonometry and Astronomy, but also in Vulgar Arithmetic, as will be manifest from Chap. IV. and especially in Annatocism, where we have sometimes occasion to extract even the three hundred sixty-fifth root of a number, as at other times to raise it to the three hundred sixty-fifth power, scarce possible to be performed any other way; to say nothing of the innumerable mistakes that in so long and laborious a calculation would be almost unavoidable, all which are prevented by the use of logarithms.

II. *Of Briggs's Logarithms.*

DEFINITION I. *A System of Logarithms* is a Table of Logarithms, in which if the common root a is determined to be a certain whole number greater than unity, the logarithms are such indexes, as are required to make the correspouding powers of that number nearly, if not exactly, equal to the natural series 1, 2, 3, 4, 5, &c. Thus, supposing $a=2$, and finding the powers of 2, which nearly, if not exactly, represent the natural series 1, 2, 3, 4, 5, &c. the indexes of these powers will be the logarithms of the natural numbers; and hence a table containing these numbers with those logarithms will be a system of logarithms.

Note. The root a must be greater than unity, because if you take $a=1$, all the powers of 1 being constantly 1, they would never become equal to another number, for instance 2.

Definition II. *Briggs's Logarithms* are a system of logarithms in which $a=10$.

Notes. I. Although all the different systems may be equally perfect, if computed to the same degree of accuracy, yet they will not all be equally convenient for use; for of all systems of logarithms, that is certainly best accommodated for practice which is now in use, and is commonly known by the name of *Briggs's Logarithms*.

II. The Lord Napeir, a Scotch nobleman was the first inventor of logarithms; but Mr. Briggs, an English gentleman, and Professor of Geometry in Gresham College, was undoubtedly the first who thought of this system.

III. The logarithms in this system are expressed by whole numbers and decimals carried to seven places, so that the integral parts of the logarithms are always distinguished from the rest, and called the *indexes* or *characteristics* of the logarithms whereof they are parts: thus the logarithm, for instance, of 20 is 1.3010300, where the characteristic is 1; that of 2 is 0.3010300, where the characteristic is 0; that of $\frac{2}{10}$ or 0.2 is $-1+.3010300$, where -1 is the characteristic, &c.

Cor. I. Taking b for any number, we shall always have in this system $10^{l.b}=b$; therefore $10^{l.1}=1$, $10^{l.2}=2$, $10^{l.3}=3$, and so on.

Cor.

Cor. II. The distinguishing mark of this system is, that herein the logarithm of 10 is 1, and consequently

$l.100=2$; $l.1000=3$; $l.10000=4$; &c.
$l.1=0$, as ought to be in every system.
$l.\frac{1}{10}=-1$; $l.\frac{1}{100}=-2$; $l.\frac{1}{1000}=-3$, &c.

Cor. III. Because the composite numbers arise from the multiplication of the primes (see pages 11 and 78) if the logarithms of prime numbers be known in this system, then by adding two, three, four, &c. of them together, the logarithms of all the composite numbers will be easily found. Thus,

$l.210=l.2+l.3+l.5+l.7$, for $210=2.3.5.7$
$l.360=3l.2+2l.3+l.5$, for $360=2.2.2.3.3.5=2^3.3^2.5$.

Cor. IV. The characteristic of the logarithm of any number is easily known, because by the progress already exposed (Cor. II.) it is manifest, that 0 is the the characteristic of numbers from 1 to 9, that 1 is the characteristic of numbers from 10 to 99, that 2 is the characteristic of numbers from 100 to 999, and so on; therefore the characteristic of any number is so many units, but one, as there are figures composing the integral part of it. If the number is a decimal fraction, its characteristic will be – the difference between the characteristics of the numerator and denominator.

Cor. V. So long as the digits that compose any number are the same, and in the same order, whatever be their places with respect to the place of units, the decimal parts of the logarithm of such a number will always be the same. The reason is, because the digits of any number multiplied or divided by 10, 100, 1000, &c. remain the same, and in the same order, and their logarithms must be altered by adding to, or subtracting from, them the logarithms of 10, 100, 1000, &c. but as these logarithms are 1, 2, 3, &c. that is whole numbers, they can only influence the characteristic of a lo-

 garithm,

garithm, without affecting the decimal part. Thus, the numbers 67.89, 678.9, 6789, 67890, &c. or 6.789, 0.6789, 0.06789, 0.006789, &c. have their logarithms with the same decimal parts, the characteristics only being different.

Notes. I. These Rules are the more to be observed, because in some tables the integral parts of all logarithms are omitted, being left to be supplied by the operator himself, as occasion requires. By this means the logarithms become of much more general use than if, by having their characteristics prefixed, they were tied down to particular numbers.

II. The common tables are sufficient for numbers, whose figures are no more than six; but the great tables of Ulacq, in which the decimal parts of the logarithms are carried to ten figures, serve also for numbers of nine figures.

SECTION V.

OF SOME OPERATIONS UPON EQUATIONS.

I. *Of the Composition of Equations, wherein of the Signs and Coefficients of their Terms.*

DEFIN. When the powers of an unknown quantity, as x, are set down according to their indexes, the term in which that quantity is of the highest power, is called the *first*; that in which its index is less by 1, is the *second*, and so on, till the last into which the unknown quan-

quantity does not enter, and which is called the *absolute* term.

PROP. I. If any number of equations be multiplied together, an equation will be produced, of which the dimension is equal to the sum of the dimensions of the equations multiplied.

Conversely. An equation of any dimension is considered as compounded either of simple equations, or of others, such that the sum of their dimensions is equal to the dimension of the given one. By the resolution of equations these inferior equations are discovered, and by investigating the component simple equations, the roots of any higher equation are found.

COR. I. Any equation admits of as many solutions or has as many roots, as there are simple equations which compose it, that is, as there are units in the dimension of it.

Cor. II. And conversely, no equation can have more roots than the units in its dimension. This may receive a further explanation. See the following Chap. III. Sect. II. Prop. Cor. VII. Note II.

Cor. III. Imaginary or impossible roots must enter an equation by pairs; for they rise from quadratics, in which both the roots are such.

Cor. IV. Hence also, an equation of an even dimension may have all its roots or any even number of them impossible, but an equation of an odd dimension must at least have one possible root.

Cor. V. The roots are either positive or negative, according as the roots of the simple equation, from which they are produced, are positive or negative.

Cor. VI. When one root of an equation is discovered, one of the simple equations is found, from which the given one is compounded. The given equation, therefore, being divided by this simple one, will give another dimension lower by 1. Thus, any equation may

may be depreſſed as many degrees as there are roots found by any method whatever.

Prop. II. To explain the general properties of the ſigns and coefficients of the terms of an equation.

Let $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, &c. be ſimple equations of which the roots are any poſitive quantities a, b, c, d, &c. and let $x+m=0$, $x+n=0$, &c. be ſimple equations, of which the roots are any negative quantities $-m$, $-n$, &c. and let any number of theſe equations be multiplied together, as in the following table:

$$\begin{array}{l}
x-a=0 \\
x-b=0 \\
\hline
\left.\begin{array}{l} x^2-ax+ab \\ \quad -bx \end{array}\right\} =0, \text{ a quadratic.} \\
x-c=0 \\
\hline
\left.\begin{array}{l} x^3-ax^2+abx-abc \\ \quad -bx^2+acx \\ \quad -cx^2+bcx \end{array}\right\} =0, \text{ a cubic.} \\
x+m=0 \\
\hline
\left.\begin{array}{l} x^4-ax^3+abx^2-abcx-abcm \\ \quad -bx^3+acx^2+abmx \\ \quad -cx^3+bcx^2+acmx \\ \quad -mx^3-amx^2+bcmx \\ \qquad\qquad -bmx^2 \\ \qquad\qquad -cmx^2 \end{array}\right\} =0, \text{ a biquadratic.}
\end{array}$$

From this table it is plain,

1. That in a complete equation the number of terms is always greater by unit than the dimenſion of the equation.

2. The coefficient of the 1ſt term is 1.

That

That of the 2d is the sum of all the roots a, b, c, m, &c. with their signs changed.

That of the 3d is the sum of all the products that can be made by multiplying any two of the roots together.

That of the 4th is the sum of all the products which can be made by multiplying together any three of the roots with their signs changed; and so of others.

The last term is the product of all the roots, with their signs changed.

3. From induction it appears, that in any equation (the terms being regularly arranged, as in the preceding example) there are as many positive roots as there are changes in the signs of the terms from + to −, and from − to +; and the remaining roots are negative. The rule also may be demonstrated.

Notes. I. The impossible roots in this rule are supposed to be either positive or negative. In this example of a numeral equation, $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, the roots are, $+1$, $+2$, $+3$, $+4$, and the preceding observations, with regard to the signs and coefficients, take place.

II. If a term of an equation is wanting, the positive and negative parts of its coefficient must then be equal. If there is no absolute term, then some of the roots must be $=0$, and the equation may be depressed by dividing all the terms by the lowest power of the unknown quantity in any of them. In this case also, $x-0=0$, $x-0=0$, $x-0=0$, &c. may be considered as so many of the component simple equations, by which, the given equation being divided, it will be depressed so many degrees.

II. Of

II. *Of the Transformation of Equations.*

There are certain transformations of equations necessary towards their solution; and the most useful are contained in the following Propositions.

PROP. I. The affirmative roots of an equation become negative, and the negative become affirmative, by changing the signs of the alternate terms, beginning with the 2d.

Thus, the roots of the equation,

$$x^4 - x^3 - 19x^2 + 49x - 30 = 0,$$

are $+1, +2, +3, -5$, whereas the roots of the equation,

$$x^4 + x^3 - 19x^2 - 49x - 30 = 0,$$

are $-1, -2, -3 + 5$.

The reason of this depends upon Cor. III. of preceding Prop. II.

Prop. II. An equation may be transformed into another that shall have its roots greater or less than the roots of the given equation, by some given difference.

Let x be the unknown quantity of the equation, and e the given difference; let $y = x \pm e$, then $x = y \mp e$; and if for x and its powers in the given equation, $y \mp e$ and its powers be inserted, a new equation will arise, in which the unknown quantity is y, and its value will be $x \pm e$; that is, its roots will differ from the roots of the given equation by $\pm e$.

Let the equation proposed be

$$x^3 - px^2 + qx - r = 0,$$

of which the roots must be diminished by e. By inserting for x and its powers, $y + e$ and its powers, the equation required is,

$$\left.\begin{array}{r} y^3 + 3ey^2 + 3eey + e^3 \\ -py^2 - 2pey - pe^2 \\ +qy + qe \\ -r \end{array}\right\} = 0.$$

COR. I.

COR. I. From this transformation, the 2d, or any other intermediate term may be taken away, granting the resolution of equations.

Since the coefficients of all the terms of the transformed equation, except the 1st, involve the powers of e and known quantities only, by putting the coefficient of any term equal to 0, and resolving that equation, a value of e may be determined; which being substituted will make that term to vanish.

Thus, in this example, to take away the 2d term, let its coefficient, $3e-p=0$, and $e=\frac{1}{3}p$, which being substituted for e, the new equation will want the 2d term. And universally, the coefficient of the 1st term of an equation of the degree m being 1, and x being the unknown quantity, the 2d term may be taken away by supposing $x=y\mp p\div m$, $\pm p$ being the coefficient of that term.

For, if the given equation be $x^m \pm px^{m-1} \ldots = 0$, then putting $x=y\mp p\div m$, the transformed equation will be

$$y^m \mp py^{m-1}, \text{\&c.} = 0.$$

$$\pm py^{m-1}, \text{\&c.} = 0.$$

in which the 2d term must vanish.

Cor. II. The 2d term may be taken away by the solution of a simple equation, the 3d by the solution of a quadratic, and so on.

Cor. III. If the 2d term of a quadratic equation be taken away, it will become a pure equation, and thus a solution of quadratics will be obtained.

Cor. IV. The last term of the transformed equation is the same with the given equation, only having e in place of x.

Prop. III. In like manner may an equation be transformed into another, of which the roots ſhall be equal to the roots of the given equation, multiplied or divided by a given quantity.

Let x be the unknown letter in the given equation, and y that of the equation wanted; alſo let e be the given quantity.

To multiply the roots let $xe=y$, and $x=y\div e$.

To divide the roots let $x\div e=y$, and $x=ye$.

Then ſubſtitute for x and its powers, $y\div e$ or ye and its powers, and the new equation of which y is the unknown quantity, will have the property required.

COR. I. By this propoſition an equation, in which the coefficient of the 1ſt term is any known quantity, as a may be transformed into another, in which the coefficient of the 1ſt term ſhall be unity. Thus, let the equation be

$$ax^3-px^2+qx-r=0.$$

Suppoſe $y=ax$, or $x=y\div a$, and for x and its powers inſert $y\div a$ and its powers, and the equation becomes

$$\frac{ay^3}{a^3}-\frac{py^2}{a^2}+\frac{qy}{a}-r=0, \text{ or}$$

$$\times a^2$$

$$y^3-py^2+qay-a^2r=0.$$

Alſo, let the equation be

$5x^3-6x^2+7x-30=0$; and if $x=y\div 5$, then
$y^3-6y^2+35y-750=0$ by ſubſtitution.

COR. II. If the two transformations in Prop. II. and III. be both required, they may be performed either ſeparately or together.

Thus,

Thus, if it is required to transform the equation

$$ax^3 - px^2 + qx - r = 0$$

into one which ſhall want the 2d term, and in which the coefficient of the 1ſt term ſhall be 1; let $x = y \div a$, and then

$$y^3 - py^2 + qay - ar^2 = 0, \text{ as before;}$$

then let $y = z + \dfrac{p}{3}$, and the new equation of which z is the unknown quantity, will want the 2d term, and the coefficient of z^3, the higheſt term, is 1. Or, if $x = \dfrac{z + \frac{1}{3}p}{a}$, the ſame equation as the laſt found, will ariſe from one operation.

Example.

Let the equation be $5x^3 - 6x^2 + 7x - 30 = 0$.

If $x = y \div 5$, then $y^3 - 6y^2 + 35y - 750 = 0$.
And if $y = z + 2$, $z^3 \quad * \quad 23z - 696 = 0$.

Alſo, at once, let $x = \dfrac{z+2}{5}$, and the equation properly reduced, by bringing all the terms to a common denominator, and then caſting it off, will be $z^3 + 23z - 696 = 0$, as before.

Cor. III. If there are fractions in an equation, they may be taken away, by multiplying the equation by the denominators, and by this Propoſition the equation may then be transformed into another, without fractions, in which the coefficient of the 1ſt term is 1. In like manner, may a ſurd coefficient be taken away in certain caſes.

Cor. IV. Hence alſo, if the coefficient of the 2d term of an equation of the degree m is not diviſible by m, the fractions thence ariſing in the transformed equation, wanting the 2d term, may be taken away by the preceding corollary. But the 2d term alſo may be taken away, ſo that there ſhall be no ſuch fractions in the transformed equation, by ſuppoſing $x = \frac{z \mp p}{m}$, $\pm p$ being the coefficient of the 2d term of the given equation. And, if, for inſtance, the cubic equation $ax^3 - px^2 + qx - r = 0$ be given, in which p is not diviſible by 3, by ſuppoſing $x = \frac{z+p}{3a}$ (and univerſally $x = \frac{z+p}{ma}$) the transformed equation reduced is

$$z^3 - \overline{3p^2 + 9aq} \times z - 2p^3 + 9apq - 27a^2r = 0,$$

wanting the 2d term, having 1 for the coefficient of the 1ſt term, and the coefficients of the other terms being all integers, the coefficients of the given equation being alſo ſuppoſed integers.

General Corollary to Propoſitions I. II. III.

If the roots of any of theſe transformed equations be found by any methods, the roots of the original equation, from which they were derived, will eaſily be found from the ſimple equations expreſſing their relation. Thus, if 8 is found to be a root of the tranſformed equation $z^3 + 23z - 696 = 0$ (Prop. III. Cor. II) ſince $x = \frac{z+2}{5}$, the correſponding root of the given equation $5x^3 - 6x^2 + 7x - 30 = 0$ muſt be $x = \frac{8+2}{5} = 2$.

CHAP.

CHAPTER III.

OF READING ALGEBRAICALLY.

DEFINITION. We are said *to read algebraically*, if, some algebraical, that is general equations being given, we may draw from them others, in which a general value of each unknown quantity is represented in known terms.

Hence our present endeavours will be the deducing final *general* equations, and resolving them.

SECTION I.

RESOLUTION OF SIMPLE GENERAL EQUATIONS.

THESE equations are resolved by the same rules belonging to the Resolution of simple numerical Equations, (See Part II. Chap. III. Sect. I.) as may be seen in the following general Examples, in which we register the several steps of the operation, in order to render it plain and evident.

Example

Example I.

Let there be these two general equations. $\begin{cases} \text{1st} & Ax+By=C \\ \text{2d} & Dx+Ey=F. \end{cases}$

$$\begin{array}{rll}
\text{Then 1st} \times E & \text{3d} & AEx+BEy=CE \\
\text{2d} \times B & \text{4th} & BDx+BEy=BF \\
\text{3d} - \text{4th} & \text{5th} & AEx-BDx=CE-BF \\
\text{5th divided by } AE-BD & \text{6th} & x=\dfrac{CE-BF}{AE-BD} \\
\text{From 1st.} & \text{7th} & y=\dfrac{C-Ax}{B}
\end{array}$$

Notes. I. The known value of x represented by the 6th equation substituted in the 7th, will give the value of y in known terms.

II. By this resolution all numerical equations containing two unknown quantities are resolved. Take, for instance,

$$5x-3y=90 \text{ and } 2x+5y=160.$$

$$\begin{array}{lll}
\text{Therefore } A= & 5\,; & CE=90\times5=450 \\
B= & -3\,; & BF=-3\times160=-480 \\
C= & 90\,; & AE=5\times5=25 \\
D= & 2\,; & BD=-3\times2=-6 \\
E= & 5\,; & CE-BF=450+480=930 \\
F= & 160\,; & AE-BD=25+6=31.
\end{array}$$

$$\text{Whence } x=\frac{CE-BF}{AE-BD}=\frac{930}{31}=30.$$

$$y=\frac{C-Ax}{B}=\frac{90-5\times30}{-3}=\frac{-60}{-3}=20.$$

Example

Example II.

Let there be these three general equations
$$\begin{cases} \text{1st} & ax+by+cz=p \\ \text{2d} & dx+ey+fz=q \\ \text{3d} & gx+my+nz=r. \end{cases}$$

$$\begin{array}{ll} \text{1st} \times f & \text{4th} \quad afx+bfy+cfz=fp \\ \text{2d} \times c & \text{5th} \quad cdx+cey+cfz=cq \\ \text{4th} - \text{5th} & \text{6th} \quad afx-cdx+bfy-cey=fp-cq \\ \text{1st} \times n & \text{7th} \quad anx+bny+cnz=np \\ \text{3d} \times c & \text{8th} \quad cgx+cmy+cnz=cr \\ \text{7th} - \text{8th} & \text{9th} \quad anx-cgx+bny-cmy=np-cr. \end{array}$$

Put $A=af-cd$; $B=bf-ce$; $C=fp-cq$;
$D=an-cg$; $E=bn-cm$; $F=np-cr$.

Then, by fubftitution, the 6th and 9th equations will become

$$Ax+By=C \text{ and } Dx+Ey=F.$$

Therefore (by Ex. I.) $x=\dfrac{CE-BF}{AE-BD}$ and $y=\dfrac{C-Ax}{B}$.

And (by the 1ft equation) $z=\dfrac{p-ax-by}{c}$.

Notes. I. The known value of x will give, by fubftitution, the value of y in known terms; then thefe two values fubftituted in the laft equation will give the value of z in known terms.

II. By this general example all numerical equations containing three unknown quantities are refolved. Suppofe, for inftance,

$$2x+y+z=68,\quad x+3y+z=102,\quad \text{and}\quad x+y+4z=136.$$

Therefore

Therefore $a = 2$,	$af = 2$;	$A = 1$
$b = 1$,	$cd = 1$,	$B = -2$
$c = 1$,	$bf = 1$,	$C = -34$
$p = 68$,	$ce = 3$,	$D = 7$
$d = 1$,	$fp = 68$,	$E = 3$
$e = 3$,	$cq = 102$,	$F = 136$
$f = 1$,	$an = 8$,	$CE = -102$
$q = 102$,	$cg = 1$,	$BF = -272$
$g = 1$,	$bn = 4$,	$AE = 3$
$m = 1$,	$cm = 1$,	$BD = -14$
$n = 4$,	$np = 272$,	$CE - BF = 170$
$r = 136$,	$er = 136$,	$AE - BD = 17$.

Whence $x = \frac{170}{17} = 10$, $y = \frac{-34-10}{-2} = \frac{-44}{-2} = 22$,

and $z = \frac{68-20-22}{1} = 26$.

III. If the number of equations and then of the unknown quantities be more, the operation increases, but the method is always the same.

SEC.

SECTION II.

RESOLUTION OF PURE GENERAL EQUATIONS.

Lemma I. $\sqrt{\frac{d-b}{a-c}} = \sqrt{\frac{b-d}{a-c}} \times \sqrt{-1}$, or

$\sqrt[r]{\frac{d-b}{a-c}} = \sqrt[r]{\frac{b-d}{a-c}} \times \sqrt[r]{-1}$.

This lemma is a corollary of multiplication of surds. Because

$\sqrt{\frac{b-d}{a-c}} \times \sqrt{-1} = \sqrt{\frac{(b-d)}{a-c} \times -1} = \sqrt{\frac{d-b}{a-c}}$.

And $\sqrt[r]{\frac{b-d}{a-c}} \times \sqrt[r]{-1} = \sqrt[r]{\frac{(b-d) \times -1}{a-c}} = \sqrt[r]{\frac{d-b}{a-c}}$.

Lemma II. $-x^2 = x^2 \sqrt{-1} \sqrt{-1}$, or in general $-x^r = x^r \times \sqrt[r]{(-1)^r}$.

Because $x^2 \sqrt{-1} \sqrt{-1} = x^2 \sqrt{-1 \times -1} = x^2 \sqrt{(-1)^2} = x^2 (-1)^{\frac{2}{2}} = x^2 \times -1 = -x^2$.

Again $x^r \times \sqrt[r]{(-1)^r} = x^r \times (-1)^{\frac{r}{r}} = x^r (-1)^1 = -x^r$.

These operations are evident from multiplication, involution, and evolution of surds.

PROP. To resolve a pure general equation.

RULE. *Make the power of the unknown quantity to stand alone on one side of the given equation, and then extract the root of the same denomination out of both sides, which will give the value of the unknown quantity.*

Example I.

Let be $ax^2+b=cx^2+d$.

Then $ax^2-cx^2=d-b$ by transposition.

$$x^2=\frac{d-b}{a-c} \text{ by division.}$$

$$-x^2=\frac{b-d}{a-c} \text{ by transposition.}$$

In the 1st case $x=\pm\sqrt{\frac{d-b}{a-c}}$ by evolution.

And $x=\pm\sqrt{\frac{b-d}{a-c}}\times\sqrt{-1}$ by the Lem. I.

In the 2d case $x^2\sqrt{-1}\sqrt{-1}=\frac{b-d}{a-c}$ by the Lem. II.

$$x\sqrt{-1}=\pm\sqrt{\frac{b-d}{a-c}} \text{ by evolution.}$$

But $x=\pm\sqrt{\frac{b-d}{a-c}}\times\sqrt{-1}$ by 1st case.

+

Then $x\times\overline{1+\sqrt{-1}}=\pm\sqrt{\frac{b-d}{a-c}}\times(1+\sqrt{-1})$ by addit.

And $x=\pm\sqrt{\frac{b-d}{a-c}}\times\frac{1+\sqrt{-1}}{1+\sqrt{-1}}$ by divis.

That is $x=\pm\sqrt{\frac{b-d}{a-c}}$.

There-

Therefore we get the two following different values of x:

1ſt, $x=\pm\sqrt{\frac{d-b}{a-c}}$; 2d, $x=\pm\sqrt{\frac{b-d}{a-c}}$.

Example II.

Let be $ax^r+b=cx^r+d$.

Then, as before, $x^r=\frac{d-b}{a-c}$, or $-x^r=\frac{b-d}{a-c}$.

In the 1ſt caſe $x=\sqrt[r]{\frac{d-b}{a-c}}$ by evolution.

And $x=\sqrt[r]{\frac{b-d}{a-c}}\times\sqrt[r]{-1}$ by the Lem. I.

In the 2d caſe $x^r\times\sqrt[r]{(-1)^r}=\frac{b-d}{a-c}$ by the Lem. II.

$x\sqrt[r]{-1}=\sqrt[r]{\frac{b-d}{a-c}}$ by evolution.

But $x=\sqrt[r]{\frac{b-d}{a-c}}\times\sqrt[r]{-1}$ by 1ſt caſe.

+

Then $x\times\overline{1+\sqrt[r]{-1}}=\sqrt[r]{\frac{b-d}{a-c}}\times(1+\sqrt[r]{-1})$ by 2d.

And $x=\sqrt[r]{\frac{b-d}{a-c}}$ by diviſion.

Hence in general, 1ſt, $x=\sqrt[r]{\frac{d-b}{a-c}}$; 2d, $x=\sqrt[r]{\frac{b-d}{a-c}}$.

Cor.

COR. I. Whatever be the index r, and the value of quantities a, b, c, d, one of the quantities under the radical sign must be positive, and the other negative (if they be not $= 0$): therefore we may always get two real roots of x, one positive, and the other negative.

Cor. II. If the index r be an odd number, one of the quantities under the radical sign will give the positive real root of x, and the other the negative.

Cor. III. If the index r be an even number, one of the quantities under the radical sign will give both the positive and negative real roots of x, and the other will give two impossible roots of it, one positive, and the other negative.

Cor. IV. If $r = 1$, then $x = \frac{d-b}{a-c}$, and $x = \frac{b-d}{a-c}$. Therefore the equations of the 1st degree have also two real roots, one positive and the other negative. This consequence may be directly demonstrated in the following manner:

Let be the root $x = a$. Then

$$x \times 1 = a \times 1, \text{ or } x \times +1 = -a \times -1$$
$$x \times -1 = a \times -1, \text{ or } x \times -1 = -a \times +1$$
$$\overline{} +$$
$$x \times 1 - 1 = -a \times 1 - 1$$
$$x = -a \times \frac{1-1}{1-1} = -a.$$

Cor. V. Every equation of the 1st degree must also have two imaginary, or impossible roots, one positive and the other negative. Because

$x = \pm a$ by Cor IV. Then

By Lemma I.

$$1 \times x = \pm 1 \times a = \pm 1 \times \sqrt{-a^2} \times \sqrt{-1}$$
$$\sqrt{-1} \times x = \pm\sqrt{-1} \times a = \pm 1 \times \sqrt{-a^2}$$

By

By Addition,

$$\begin{array}{ll} 1\times x & \pm 1\times\sqrt{-a^2}\times\sqrt{-1} \\ \sqrt{-1}\times x & \pm 1\times\sqrt{-a^2} \\ \hline (1+\sqrt{-1})\times x \quad + \quad = & \pm\sqrt{-a^2}\times(1+\sqrt{-1}) \quad + \end{array}$$

And $x=\pm a\sqrt{-1}$ by Division and Evolution.

Cor. VI. Hence every equation of the 1st degree has 4 roots, two real, and two imaginary, one positive and one negative of each sort.

Cor. VII. And because the superior equations arise from the multiplication of inferiors, any equation admits of four times as many roots, as there are simple equations which compose it: of these roots half are real, half imaginary; and of both real, and imaginary, half are positive, and half negative. For instance, an equation of the 3d order has 12 roots, 6 real, and 6 imaginary, and of each sort 3 positive, and 3 negative.

Note I. The index of the power may also be fractional, as in this equation, $x^{m\div n}=\frac{d-b}{a-c}$.

In this case, raise the equation to the power n, and it will be $x^m=\left(\frac{d-b}{a-c}\right)^n$; then by Evolution

$$x=\sqrt[m]{\left(\frac{d-b}{a-c}\right)^n}, \text{ or } x=\left(\frac{d-b}{a-c}\right)^{n\div m}.$$

Examples.

I. Let $ax^{\frac{1}{2}}-b=x^{\frac{1}{2}}-c$. Then $m=1$, $n=2$,

$$x=\sqrt[1]{\left(\frac{d-b}{a-c}\right)^2}=\left(\frac{d-b}{a-c}\right)^2=\frac{d^2-2bd+bb}{a^2-2ac+c^2}$$

II. Let

II. Let $ax^{\frac{3}{2}} - b = x^{\frac{3}{2}} - c$. Then $m = 3$, $n = 2$.

$$x = \sqrt[3]{\left(\frac{d-b}{a-c}\right)^2} = \sqrt[3]{\frac{d^2 - 2bd + bb}{a^2 - 2ac + c^2}}.$$

Note II. As this number of roots is a new discovery in Algebra, so to avoid every confusion, which may arise from this new doctrine, it will be convenient to distinguish all the roots of an equation into *primitive*, which are found by the common way, and *derivative*, which arise from our new method of investigating them. Therefore what we said upon the roots of an equation in Sect. V. of the preceding Chap. II. must be understood of the primitives. The derivative roots shew the generality and fecundity of the algebraical language, and each real root, if it does not give a direct and proper answer to the question, from which such an equation has been derived, will always, if well understood, open a real meaning of the question itself under a different point of view.

SECTION III.

RESOLUTION OF GENERAL QUADRATIC EQUATIONS.

LEMMA. $\sqrt{(a^2 - b)^2} = \pm(a^2 - b)$.

Put $\pm(a^2 - b) = \pm r$. Then by Involution $(a^2 - b)^2 = r^2$, and by Evolution $\sqrt{(a^2 - b)^2} = \pm r = \pm(a^2 - b)$.

In the same manner we get $\sqrt{(b - a^2)^2} = \pm(b - a^2)$.

PROP. To resolve the general quadratic equation
$$x^2+2ax+b=0.$$

Put $x=m+n\sqrt{-1}$. Then $x-m=n\sqrt{-1}$;

$$\left.\begin{array}{ll}\text{And} & x^2-2mx+m^2=-n^2 \\ \text{Or} & x^2-2mx+m^2=(n\sqrt{-1})^2\end{array}\right\}\text{ by Involution.}$$

Hence $x^2-2mx+m^2+n^2=0$

Or $x^2-2mx+m^2-(n\sqrt{-1})^2=0$

But $x^2+2ax+b=0$ by supposition.

Therefore, by the laws of equations, $m=-a$;

$$\begin{array}{llll} & n^2+m^2=b & & -(n\sqrt{-1})^2+m^2=b \\ \text{Or} & n^2=b-a^2 & \text{Or} & (n\sqrt{-1})^2=a^2-b \\ & n=\pm\sqrt{(b-a^2)} & & n\sqrt{-1}=\pm\sqrt{(a^2-b)}\text{ 1st val.}\end{array}$$

Also $n=\pm\sqrt{(a^2-b)}\times\sqrt{-1}$.

But $n\sqrt{-1}=\pm\sqrt{(a^2-b)}$

———————————— +

Then $n\times(1+\sqrt{-1})=\pm\sqrt{(a^2-b)}\times(1+\sqrt{-1})$

$n=\pm\sqrt{(a^2-b)}$ by dividing by $(1+\sqrt{-1})$

$n\sqrt{-1}=\pm\sqrt{(a^2-b)}\times\sqrt{-1}$ by multiplying by $\sqrt{-1}$

Whence we draw $n\sqrt{-1}=\pm\sqrt{(b-a^2)}$ 2d value.

Therefore being $x=m+n\sqrt{-1}$ by supposition,

We get, 1st, $x=-a\pm\sqrt{(a^2-b)}$.

2d, $x=-a\pm\sqrt{(b-a^2)}$.

Note I. Both these values satisfy the general equation $x^2+2ax+b=0$.

Because, 1st, $x^2=a^2\mp2a\sqrt{(a^2-b)}+\sqrt{(a^2-b)}^2$

$2ax=-2a^2\pm2a\sqrt{(a^2-b)}$

$b=b$.

2d, $x^2=a^2\mp2a\sqrt{(b-a^2)}+\sqrt{(b-a^2)}^2$

$2ax=-2a^2\pm2a\sqrt{(b-a^2)}$

$b=b$.

$$\left.\begin{array}{ll}\text{But} & \sqrt{(a^2-b)}^2=\pm(a^2-b) \\ \text{And} & \sqrt{(b-a^2)}^2=\pm(b-a^2)\end{array}\right\}\text{ by the Lemma.}$$

Therefore taking $+(a^2-b)$ in the 1st case.

$-(b-a^2)$ in the 2d.

The sum of the equations shall always be $0=0$.

Note

Note II. An equation, in the terms of which two powers only of the unknown quantity are found, and such that the index of the one is double that of the other, may, by this Proposition, be reduced to a pure equation, and may therefore be resolved by the preceding Section. Such an equation may generally be represented thus:

$$x^{2m} + 2ax^{m} + b = 0.$$

Let $x^m = z$, then $z^2 + 2az + b = 0$;

$$\left.\begin{array}{l} \text{And } z = -a \pm \sqrt{(a^2 - b)} \\ \text{Or } \; z = -a \pm \sqrt{(b - a^2)} \end{array}\right\} \text{ by the Proposition.}$$

Therefore, being $x = \sqrt[m]{z}$ by supposition,

We have $x = \sqrt[m]{(-a \pm \sqrt{a^2 - b})}$,

and $x = \sqrt[m]{(-a \pm \sqrt{b - a^2})}$.

SECTION IV.

RESOLUTION OF GENERAL CUBIC EQUATIONS.

LEMMA. $\dfrac{d^3}{e \pm \sqrt{e^2 - d^3}} = e \mp \sqrt{e^2 - d^3}$.

For $\dfrac{d^3}{e \pm \sqrt{e^2 - d^3}} = \dfrac{d^3 \times (e \mp \sqrt{e^2 - d^3})}{(e \pm \sqrt{e^2 - d^3}) \times (e \mp \sqrt{e^2 - d^3})} =$

$\dfrac{d^3 \times (e \mp \sqrt{e^2 - d^3})}{e^2 - e^2 + d^3} = e \mp \sqrt{e^2 - d^3}$.

PROP.

PROP. To resolve the general cubic equation

$$x^3+3ax^2+3bx+2c=0.$$

[We put in the equation some numerical coefficients, in order to avoid the fractions in the following operations.]

Suppose $x=m+n-p$; then $x+p=m+n$; and by raising to the third power,

$$x^3+3px^2+3p^2x+p^3=m^3+3nm^2+3n^2m+n^3$$
(by Decomposition) $=m^3+3mn\times\overline{m+n}+n^3$
(by Substitution) $=m^3+3mn\times\overline{x+p}+n^3$
(by Multiplication) $=m^3+3mnx+3mnp+n^3$.

Hence by Transposition we get the artificial equation

$$\begin{aligned} x^3+3px^2+3p^2x&+p^3=0\\ -3mnx&-3mnp\\ &-m^3\\ &-n^3 \end{aligned}$$

But $x^3+3ax^2+3bx+2c=0$.

Therefore $3p=3a$, or $p=a$;
$3p^2-3mn=3b$, or $mn=p^2-b$;
$p^3-3mnp-m^3-n^3=2c$, or
$m^3+n^3=p^3-3mnp-2c$.

Now $\left.\begin{aligned} mn&=p^2-b=a^2-b=d\\ p^3-3mnp-2c&=a^3-3ad-2c=2e \end{aligned}\right\}$ for simplicity's sake.

Then $m^3+n^3=2e$, and $m^3n^3=d^3$

$\times m^3$

$m^6+m^3n^3=2em^3$
$m^3n^3=d^3$

$-$

$m^6 \quad * \quad =2em^3-d^3$, or
$m^6-2em^3+d^3=0$.

Whence $m=\sqrt[3]{e\pm\sqrt{e^2-d^3}}$ by Sect. III. Not. II.

And $n^3=\dfrac{d^3}{m^3}=\dfrac{d^3}{e\pm\sqrt{e^2-d^3}}=e\mp\sqrt{e^2-d^3}$ by the Lem.

 Then

Then $n=\sqrt[3]{e\mp\sqrt{e^2-d^3}}$ by Evolution.

And $x=m+n-p$ by supposition. Therefore

$$x=-a+\sqrt[3]{e+\sqrt{e^2-d^3}}+\sqrt[3]{e-\sqrt{e^2-d^3}}.$$

If you take the other value, $m=\sqrt[3]{e+\sqrt{d^3-e^2}}$, according to our method (Not. II. Sect. III.), you will find the following value:

$$x=-a+\sqrt[3]{e+\sqrt{d^3-e^2}}+\sqrt[3]{e-\sqrt{d^3-e^2}}.$$

COR. I. If $3a=0$, or $x^3+3bx+2c=0$, then

$$1\text{st},\ x=\sqrt[3]{-c+\sqrt{c^2+b^3}}+\sqrt[3]{-c-\sqrt{c^2+b^3}},$$

$$\&\ 2\text{d},\ x=\sqrt[3]{-c+\sqrt{-c^2-b^3}}+\sqrt[3]{-c-\sqrt{-c^2-b^3}}.$$

Note. The 1st value of x is Cardan's canon for cubic equations. And it is to be remarked, that if in the given equation $3b$ is negative, and if c^2 is less than b^3, this expression of the root involves impossible roots; while, at the same time, all the roots of that equation are possible. This is called the *irreducible case*, to which belongs, for instance, the equation $x^3-156x+560=0$, whose roots are $+4$, $+10$, -14. If the cube root of the compound surd can be extracted, the impossible parts balance each other, and the true root is obtained. But the other value of x, found by our method, is a possible root in this case, and answers to the general equation $x^3+3bx+2c=0$. See the Preface.

Cor. II. If $3b=0$, or $x^3+3ax^2+2c=0$, then

$$1\text{st},\ x=-a+\sqrt[3]{-a^3-c+\sqrt{2a^3c+c^2}}+\sqrt[3]{-a^3-c-\sqrt{2a^3c+c^2}},$$

$$2\text{d},\ x=-a+\sqrt[3]{-a^3-c+\sqrt{-2a^3c-c^2}}+\sqrt[3]{-a^3-c-\sqrt{-2a^3c-c^2}}.$$

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SECTION V.

RESOLUTION OF GENERAL BIQUADRATIC EQUATIONS.

PROP. To resolve the general biquadratic equation

$$x^4+2ax^3+bx^2+2cx+d=0,$$

Take $\quad x^2+px+q=mx+n.$

Raise this equation to the 2d power, and then, by Transposition, you will get the biquadratic equation

$$\begin{array}{llll} x^4+2px^3 & +p^2x^2 & +2pqx & +q^2=0 \\ & 2qx^2 & & \\ & -m^2x^2 & -2mnx & -n^2. \end{array}$$

But $\quad x^4+2ax^3+bx^2+2cx+d=0.$

Therefore $p=a$; $p^2+2q-m^2=b$; $pq-mn=c$; $q^2-n^2=d$.

That is $m^2=2q+a^2-b$; $mn=aq-c$; $n^2=q^2-d$.

Hence $m^2n^2=\overline{2q+a^2-b}\times\overline{q^2-d}$, and $m^2n^2=\overline{aq-c}^2$,

$$\begin{array}{lllll} \text{Or} & m^2n^2=2q^3+a^2q^2 & -2dq & -a^2d \\ & \qquad\qquad -bq^2 & & +bd \\ \text{And} & m^2n^2= \qquad a^2q^2 & -2acq & +c^2 \\ \hline & 2q^3-bq^2 & -2dq & -a^2d=0 \\ & & 2acq & +bd \\ & & & -c^2 \end{array}$$

 Sup-

Suppose $q=\frac{z}{2}$, and by inserting $\frac{z}{2}$ instead of q, then by multiplying by 4 you will transform this cubic equation in the following:

$$\begin{array}{llll} z^3 - bz^2 & -4dz & -4a^2d & =0 \\ & 4acz & +4bd & \\ & & -4c^2. & \end{array}$$

This equation, resolved by Sect. IV. will give a value of z; hence it will be easy to get the values of q, m, and n, which substituted in the arbitrary equation $x^2+px+q=mx+n$, will give a quadratic equation, by which the general biquadratic being divided; the quotient will be another quadratic equation; and the roots of these two equations will be also the root of the general biquadratic itself, which is the product of them both multiplied together.

Note. There are no methods to resolve the *general* equations of higher orders; but you may sometimes *reduce*, that is, *resolve*, an higher *numerical* equation into its inferior components, as will appear from the following Section.

SECTION VI.

RESOLUTION OF HIGHER NUMERICAL EQUATIONS, BY THEIR LOWER COMPONENTS.

PROP. I. To resolve the numerical equation

$$x^5+5x^4+4x^3-4x^2+6x+3=0,$$

Because

Becaufe 3 is the product of 3×1, fuppofe

$$\begin{array}{r} x^3+ax^2+\ bx\ +3=0 \\ x^2+\ cx\ +1=0 \\ \hline x^5+ax^4+bx^3+3x^2+3cx+3=0 \\ cx^4+acx^3+bcx^2+bx \\ x^3+ax^2 \end{array} \times$$

Then $a+c=5$; $ac+b+1=4$; $a+bc+3=-4$; $b+3c=6$.

Hence $b=3-ac$, and $b=6-3c$. Therefore $3-ac=6-3c$, and $3c-ac=3$. Whence

$c=\frac{3}{3-a}$. But alfo $c=5-a$. Therefore

$5-a=\frac{3}{3-a}$, or $15-3a-5a+a^2=3$. That is $a^2-8a+12=0$, whofe roots $a=4\pm 2$.

Take $a=6$; then $c=5-6=-1$; $ac=-6$; $b=3-ac=3+6=9$; $bc=-9$. But $a+bc+3=-4$. Therefore $6-9+3=-4$, that is $0=-4$, which is impoffible.

Whence we muft take $a=2$; then $c=3$; $ac=6$; $b=3-ac=3-6=-3$; $bc=-9$; and $a+bc+3=2-9+3=5-9=-4$, as muft be.

The two component equations are therefore

$$\begin{array}{r} x^3+2x^2-3x+3\ =0 \\ x^2+3x+1\ =0 \\ \hline x^5+2x^4-3x^3+3x^2+9x+3=0 \\ 3x^4+6x^3-9x^2-3x \\ x^3+2x^2 \\ \hline x^5+5x^4+4x^3-4x^2+6x+3=0. \end{array} \times$$

Note.

Note. If you take the following component equations,

$$x^3+ax^2+bx+1=0$$
$$x^2+cx+3=0$$

you will find that they do not anſwer your purpoſe.

Hence you ſee, that the whole matter depends upon trials; but, if the propoſed equation is *reducible*, you will always ſucceed.

Prop. II. To reſolve the numerical equation

$$x^6 * -7x^4+8x^3 * +2x+1=0.$$

$$\begin{array}{l} \text{Take } x^3+ax^2+bx+1=0 \\ \quad\quad\; x^3-ax^2+cx+1=0 \\ \hline x^6+ax^5+bx^4+x^3-ax^2+cx+1=0 \\ \quad -ax^5-a^2x^4-abx^3+bcx^2+bx \\ \quad\quad cx^4+acx^3+ax^2 \\ \quad\quad\quad x^3 \\ \hline x^6 \quad * \quad +bx^4+2x^3+bcx^2+cx+1=0 \\ \quad\quad -a^2x^4-abx^3 \quad\quad +bx \\ \quad\quad cx^4+acx^3 \end{array} \begin{array}{l} \\ \\ \times \\ \\ \\ \\ + \end{array}$$

Hence $b+c=a^2-7$; $ac-ab=6$; $bc=0$; $b+c=2$;
Therefore $a^2-7=2$, or $a^2=9$, and $a=\pm 3$;
Now take $a=3$, $b=0$, it will be $c=2$; and
$ac-ab=6-0=6$, as muſt be.

Wherefore the component equations are

$$\begin{array}{l} x^3+3x^2 \quad * \quad +1=0 \\ x^3-3x^2+2x+1=0 \\ \hline x^6+3x^5 \quad * \quad +x^3-3x^2+2x+1=0 \\ \quad -3x^5-9x^4+6x^3+3x^2 \\ \quad\quad 2x^4+x^3 \\ \hline x^6 \quad * \quad -7x^4+8x^3 \quad * \quad +2x+1=0 \end{array} \begin{array}{l} \\ \\ \times \\ \\ \\ + \end{array}$$

Note.

Note. If we could not succeed with two component cubic equations, we should try two others, one of the 4th, and the other of the 2d degree; or we should take three quadratic equations. But, if all our trials are without success, the proposed equation becomes *irreducible*, and its roots must be found by *approximation*. See Part II. Chap. III. Sect. II.

SECTION VII.

RESOLUTION OF EQUATIONS BY CONVERGING SERIES.

WHEN an equation may not be resolved by the preceding methods, the best way of finding its roots is that of converging series, as universal, extending to all kinds of equations, and though not accurately true, it gives the value sought with little trouble, to a very great degree of exactness. It is the same method, which we explained in Part II. Chap. III. Sect. II. Division II.; but here it is applied to a general equation of any degree whatsoever.

PROP. To resolve, by converging series, the general equation

$$ax^{m}+bx^{m-1}+cx^{m-2}+dx^{m-3}+ex^{m-4}+, \text{ \&c. } =Q.$$

Let $x=r+z$, r being the root nearly estimated, and z the difference between that value, and the true value x. Then substituting, instead of x, in the given equation, its supposed value $r+z$, and neglecting the powers

powers z^2, z^3, &c. there will emerge the transformed equation

$$\left.\begin{array}{l} ar^m + mar^{m-1}z + br^{m-1} + \overline{m-1} \times br^{m-2}z + cr^{m-2} \\ \quad + \overline{m-2} \times cr^{m-3}z + dr^{m-3} + \overline{m-3} \times dr^{m-4}z \\ \quad + er^{m-4} + \overline{m-4} \times er^{m-5}z + \ldots\ldots \end{array}\right\} = Q.$$

From which is found

$$z = \frac{Q - ar^m - br^{m-1} - cr^{m-2} - dr^{m-3} - er^{m-4} -, \&c.}{mar^{m-1} + \overline{m-1} \times br^{m-2} + \overline{m-2} \times cr^{m-3} + \overline{m-3} \times dr^{m-4} +, \&c.}$$

As an inſtance of the uſe of this theorem, let the equation

$$-x^3 + 300x = 1000;$$

in which caſe $m=3$, $a=-1$, $b=0$, $c=300$, and $Q=1000$; by ſubſtituting theſe values, we ſhall have

$$z = \frac{1000 + r^3 - 300r}{-3r^2 + 300};$$

where (ſince it appears, by inſpection, that one of the values of x muſt be greater than 3, but leſs than 4) let r be taken $=3$, and then will z become $= \frac{127}{273} = 0.5$, and conſequently

$x=r+z=3.5$, nearly. Therefore, in order to repeat the operation, let 3.5 be writ instead of r, in the preceding equation, and z will come out $=\frac{-7.125}{263.25}$ $=-0.027$, which added to 3.5, gives 3.473, for the value of x, more nearly. And, by repeating the operation once more, x will be found $=3.47296351$, which is true to 7 or 8 places of Decimals.

COR. If the root of a pure power is to be extracted, or, which is the same, if the proposed equation be $x^m=Q$, then a being $=1$, and b, c, d, &c. each $=0$, z in this case will be barely $=\frac{Q-r^m}{mr^{m-1}}$; which may serve as a general theorem for extracting the roots of pure powers. Thus, if it were required to extract the cube root of 10, them m being $=3$ and $Q=10$, z will be $=\frac{10-r^3}{3r^2}$; in which let r be taken $=2$, and it will become $z=\frac{2}{12}=0.16$. Therefore $x=2.16$; from which, by repeating the operation, the next value of x will be found $=2.1544$.

SECTION VIII.

METHOD OF DEDUCING FINAL GENERAL EQUATIONS.

RULE. *Consider every power of one unknown quantity, as an unknown quantity itself; then, by the Rules given for simple general equations (see Chap.* III. *Sect.* I.*), you will get a final general equation, including only one unknown quantity,*

 Example,

Example.

$$\left.\begin{array}{l}\text{Let 1st, } ax^2 + by^2 + cxy + dx + ey = f \\ \text{And 2d, } gx^2 + by^2 + kxy + lx + my = n\end{array}\right\} \times \begin{array}{l} g \\ a \end{array}$$

$$\begin{array}{l} agx^2 + bgy^2 + cgxy + dgx + egy = fg \\ agx^2 + aby^2 + akxy + alx + amy = an \end{array}$$

$$\overline{bg - ab} \times y^2 + \overline{\begin{array}{c} cgy - aky \\ dg - al \end{array}} \times x + \overline{eg - am} \times y = fg - an.$$

Take $A = bg - ab$; $B = cgy - aky + dg - al$; $C = eg - am$; $D = fg - an$; and you will have the following equation:

$$Ay^2 + Bx + Cy = D$$
$$\times x$$
$$Ay^2x + Bx^2 + Cyx = Dx, \text{ or}$$

$$Bx^2 + \overline{Ay^2 + Cy - D} \times x = 0$$
$$\times a$$
$$Bax^2 + \overline{Aay^2 + Cay - Da} \times x = 0$$

$$1\text{st} \times B \quad Bax^2 + Bby^2 + \overline{Bcy + Bd} \times x + Bey = Bf$$

$$Bby^2 + \overline{Bcy + Bd - Aay^2 - Cay + Da} \times x + Bey = Bf,$$

Hence $x = \dfrac{Bf - Bby^2 - Bey}{Bcy + Bd - Aay^2 - Cay + Da}$.

But also $x = \dfrac{D - Ay^2 - Cy}{B}$. Therefore

$$\frac{D - Ay^2 - Cy}{B} = \frac{Bf - Bby^2 - Bey}{Bcy + Bd - Aay^2 - Cay + Da}.$$

Which equation contains only the unknown quantity y; and being cleared from fractions will offer it raised to the 4th power.

Note.

Note. This general example will be sufficient for our purpose, which was only to explain a method, whose use is uncommon in Algebra. But there are sometimes equations, which, worked by this method, would give a final equation of a very high degree, which by their nature may be very easily resolved, as will appear from the following example:

There are three numbers in a continual geometrical proportion, their sum is 74, and the sum of their squares is 1924: what are those numbers?

Let the three numbers be x, y, and z. Therefore

$$\begin{array}{ll}
\text{1st, } x:y::y:z, \text{ or} & xz=y^2; \\
\text{2d, } x+y+z=74, \text{ or} & x+z=74-y; \\
\text{3d, } x^2+y^2+z^2=1924, \text{ or} & \left.\begin{array}{l} x^2+z^2=1924-y^2 \\ 2xz=2y^2 \end{array}\right\} \\
\hline
& x^2+2xz+z^2=1924+y^2 \quad + \\
& x^2+2xz+z^2=5476-148y+y^2 \\
\hline
& 0=3552-148y \\
\hline
& 0=24-y. \quad \div 148
\end{array}$$

Whence $y=24$, $xz=576$, and $x+z=50$

$$\begin{array}{l}
x+z=50 \\
\hline
x^2+xz=50x \quad \times x \\
xz=576 \\
\hline
x^2 \quad * \quad =50x-576.
\end{array}$$

The final equation is therefore only of the 2d degree, and being resolved, gives $x=18$, and hence $z=32$.

CHAPTER III.

OF SPEAKING ALGEBRAICALLY.

DEFINITION. WE are ſaid to *ſpeak algebraically*; 1ſt, when we reaſon abſtractedly upon our problems, and propoſing them indefinitely we give an indefinite ſolution, from which a general Rule or Theorem may be drawn, for ſolving all particular caſes to which they are applicable; and 2d, when we demonſtrate Theorems with regard to all thoſe quantities concerning which the algebraical language may be uſed as an analyſis.

Note. Such a general inveſtigation of the unknown quantity, is called the *Analyſis*, or *Analytical inveſtigation* of the problem. The aſſuming the value of the unknown quantity (in known terms), ſuch as the problem finally determines that value to be; and ſhewing that the quantities ſo aſſumed have the properties deſcribed in the problem, is called the *Synthetical Demonſtration*. The tranſlation of the unknown quantity (in known terms) out of the algebraic into common language, is deducing a *Theorem*, or forming a *Canon*, for all caſes of the like kind, according to the form of words in which the tranſlation is made. If the equality between the unknown quantity and its value (in known terms) be ſimply declared; it is a theorem. If the arithmetical operations for computing the value of the unknown quantity (as algebraically expreſſed) be laid down in words at length, this is called a Canon, or Rule for computing the value of the quantity ſought in all queſtions of

of the like kind. Of all this we shall give one complete example; though all problems of the next Section I. must be, not only analytically investigated, but synthetically demonstrated.

SECTION I.

GENERAL SOLUTION OF PROBLEMS.

LET x and y be any two numbers, whereof x is the greater and y the less.

Let their sum	$x+y=s$,
their difference	$x-y=d$,
their product	$xy=p$,
their quotient	$x\div y=q$,
the sum of their squares	$x^2+y^2=a$,
the difference of their squares	$x^2-y^2=b$.

Then any two of these six being given, the numbers may be found; or any of the quantities without finding the numbers themselves, as may appear from the following 12 problems:

PROBL. I. Given s and d, quære x and y.

$$\begin{array}{ll} x+y=s & x+y=s \\ x-y=d & x-y=d \\ \hline 2x \quad * = s+d & * \quad 2y=s-d \\ x=\dfrac{s+d}{2}. & y=\dfrac{s-d}{2}. \end{array}$$

Synthetical Demonstration. $\dfrac{s+d}{2}+\dfrac{s-d}{2}=\dfrac{2s}{2}=s.$

Again,

Again, $\frac{s+d}{2}-\frac{s-d}{2}=\frac{2d}{2}=d.$

Hence THEOREM I. *The difference between any two numbers, added to their sum, is equal to twice the greater.*

Theorem II. *The difference between any two numbers, subtracted from their sum, is equal to twice the less.*

CANON. *Add the difference to the sum, and half the aggregate will be the greater number.* Again. *Subtract the difference from the sum and half the remainder will be the less number.*

Example.

Let their sum be 19, and their difference 10: Here

$\frac{s+d}{2}=\frac{29}{2}=14\frac{1}{2}$, the greater number; and

$\frac{s-d}{2}=\frac{9}{2}=4\frac{1}{2}$, the less. For

$14\frac{1}{2}+4\frac{1}{2}=19$, and $14\frac{1}{2}-4\frac{1}{2}=10$, as required.

Note. We are at liberty to change the form of the general algebraic expressions, which determine x and y, provided we do not change their value. Thus we may separate the two members which compose the numerators of the fractional values of x and y, and they will stand in this form: $x=\frac{s}{2}+\frac{d}{2}$, and $y=\frac{s}{2}-\frac{d}{2}$. This will afford great variety of expression in drawing out a theorem or canon. It may now stand thus: to the semi-sum of the two numbers add their semi-difference, and it makes the greater; and from the semi-sum subtract their semi-difference, and it leaves the less number.

PROBL.

PROBL. II. Given s and b, quære x and y?

$$x+y=s; \quad x=s-y; \quad xx=ss-2sy+yy$$
$$xx-yy=b; \quad \text{and} \quad xx=b \quad +yy$$
$$0=ss-2sy-b$$

Hence $y=\frac{ss-b}{2s}$, and $x=s\frac{-ss+b}{2s}=\frac{ss+b}{2s}$.

Synthesis. $\frac{ss+b}{2s}+\frac{ss-b}{2s}=\frac{2ss}{2s}=s$;

$$\left(\frac{ss+b}{2s}\right)^2-\left(\frac{ss-b}{2s}\right)^2=\frac{s^4+2bss+bb}{4ss}-\frac{s^4+2bss-bb}{4ss}=b.$$

PROBL. III. Given d and b, quære x and y?

$$x-y=d; \quad x=d+y; \quad xx=dd+2dy+yy$$
$$xx-yy=b; \quad \text{and} \quad xx=b \quad +yy$$
$$0=dd+2dy-b$$

Hence $y=\frac{b-dd}{2d}$, and $x=d+\frac{b-dd}{2d}=\frac{b+dd}{2d}$.

Synthesis. $\frac{b+dd}{2d}+\frac{b+dd}{2d}=\frac{2dd}{2d}=d$;

$$\left(\frac{b+dd}{2d}\right)^2-\left(\frac{b-dd}{2d}\right)^2=\frac{bb+2bdd+d^4}{4dd}-\frac{bb+2bdd-d^4}{4dd}$$
$$=b.$$

Note. Instead of $\frac{b+dd}{2d}$ and, $\frac{b-dd}{2d}$, we may put $\frac{b}{2d}+\frac{d}{2}$, and $\frac{b}{2d}-\frac{d}{2}$, then draw a canon from either expression.

PROBL.

PROBL. IV. Given s and q, quære x and y?

$$x+y=s; \quad x=s-y;$$
$$\frac{x}{y}=q; \quad x=qy$$
$$0=qy+y-s.$$

Hence $y=\frac{s}{q+1}$, and $x=s\frac{-s}{q+1}=\frac{qs}{q+1}$.

Synthesis. $\frac{qs}{q+1}+\frac{s}{q+1}=\frac{s\times\overline{q+1}}{q+1}=s$,

$\frac{qs}{q+1}\div\frac{s}{q+1}=\frac{qs}{q+1}\times\frac{q+1}{s}=q$.

PROBL. V. Given d and q, quære $x+y$?

$$x-y=d; \quad x=d+y;$$
$$\frac{x}{y}=q; \quad x=qy$$
$$0=qy-y-d.$$

Hence $y=\frac{d}{q-1}$, and $x=qy=\frac{dq}{q-1}$.

Synthesis. $\frac{dq}{q-1}\,\frac{-d}{q-1}=\frac{d\times\overline{q-1}}{q-1}=d$,

$\frac{dq}{q-1}\div\frac{d}{q-1}=\frac{dq}{q-1}\times\frac{q-1}{d}=q$.

PROBL. VI. Given p and q, quære x and y?

$$xy=p; \quad x=p\div y$$
$$\frac{x}{y}=q; \quad x=qy$$
$$0=qy-p\div y.$$

Hence

Hence $qy=\frac{p}{y}$; $yy=\frac{p}{q}$, and $y=\sqrt{\frac{p}{q}}$.

Again, $x=qy=q\sqrt{\frac{p}{q}}=\sqrt{\frac{pqq}{q}}=\sqrt{pq}$.

Synthesis. $\sqrt{pq}\times\sqrt{\frac{p}{q}}=\sqrt{\frac{ppq}{q}}=\sqrt{pp}=p$.

$\sqrt{pq}\div\sqrt{\frac{p}{q}}=\sqrt{pq}\div\frac{p}{q}=\sqrt{pq}\times\frac{q}{p}=\sqrt{qq}=q$.

PROBL. VII. Given s and p, quære x and y?

$$x+y=s; \qquad x=s-y;$$
$$xy=p; \qquad x=p\div y$$
$$0=\frac{p}{y}+y-s.$$

Hence $yy-sy+p=0$, and putting $\frac{RR}{4}=\frac{ss}{4}-p$,

$y=\frac{s\pm R}{2}$ (by Chap. III. Sect. III.), and

$x=s\,\frac{-s\mp R}{2}=\frac{s\mp R}{2}$.

Becauſe, by ſuppoſition, x is the greater, and y the leſs number, we ſhall have $x=\frac{s+R}{2}$, and $y=\frac{s-R}{2}$.

Synthesis. $\frac{s+R}{2}+\frac{s-R}{2}=\frac{2s}{2}=s$.

$\frac{s+R}{2}\times\frac{s-R}{2}=\frac{ss-RR}{4}=\frac{ss}{4}-\frac{ss}{4}+p=p$.

 PROBL.

PROBL. VIII. Given d and p, quære x and y.

$$x-y=d;\ x=y+d;$$
$$xy=p;\ \underline{x=p\div y}$$
$$0=y+d-\frac{p}{y}.$$

Hence $y^2+dy-p=0$, and putting $\frac{RR}{4}=\frac{dd}{4}+p$,

$y=-\frac{d\pm R}{2}$ (by Chap. III. Sect. III.) and

$x=d\frac{-d\pm R}{2}=\frac{d\pm R}{2}$.

Synthesis. $\frac{d\pm R}{2}+\frac{d\mp R}{2}=\frac{2d}{2}=d.$

$$\frac{d\pm R}{2}\times\frac{-d\pm R}{2}=\frac{-dd+RR}{4}=\frac{-dd}{4}+\frac{dd}{4}+p=p.$$

PROBL. IX. Given s and a, quære x and y.

$$x+y=s;\ x=s-y;\ xx=ss-2sy+yy;$$
$$xx+yy=a;\ \text{and}\ \underline{xx=a\ \ \ -yy}$$
$$0=ss-a-2sy+2yy.$$

Hence $yy-sy+\frac{ss-a}{2}=0$, and putting $\frac{RR}{4}=\frac{a}{2}-\frac{ss}{4}$,

$y=\frac{s\pm R}{2}$ (by Chap. III. Sect. III.) and

$x=s\frac{-s\mp R}{2}=\frac{s\mp R}{2}$.

$x=\frac{s+R}{2}$ for the greater, and $y=\frac{s-R}{2}$ for the less number.

Synthesis.

Synthesis. $\frac{s+R}{2}+\frac{s-R}{2}=\frac{2s}{2}=s.$ Again

$$\left(\frac{s+R}{2}\right)^2+\left(\frac{s-R}{2}\right)^2=\frac{ss+2Rs+RR}{4}+\frac{ss-2Rs+RR}{4}$$
$$=\frac{2ss+2RR}{4}=\frac{2ss}{4}+a-\frac{2ss}{4}=a.$$

PROBL. X. Given d and a, quære x and y.

$$\begin{array}{ll} x-y=d;\ x=d+y; & xx=dd+2dy+yy \\ xx+yy=a;\ \text{and} & xx=a\qquad\quad -yy \\ \hline & 0=dd-a+2dy+2yy. \end{array}$$

Hence $yy+dy+\frac{dd-a}{2}=0$; and putting $\frac{RR}{4}=\frac{a}{2}-\frac{dd}{4}$,

$y=\frac{-d\pm R}{2}$ (by Ch. III. Sect. III.) and

$x=d\frac{-d\pm R}{2}=\frac{d\pm R}{2}.$

Synthesis. $\frac{d\pm R}{2}+\frac{d\mp R}{2}=\frac{2d}{2}=d.$ Again

$$\left(\frac{d\pm R}{2}\right)^2+\left(\frac{-d\pm R}{2}\right)^2=\frac{d^2\pm 2dR+R^2}{4}+\frac{d^2\mp 2dR+R^2}{4}$$
$$=\frac{2d^2+2R^2}{4}=\frac{2d^2}{4}+a-\frac{2d^2}{4}=a.$$

PROBL. XI. Given p and a quære x and y.

$$xy=p\,;\ x=p\div y\,;\ xx=pp\div y^2$$
$$xx+yy=a\,;\ \text{and}\qquad xx=a-yy$$
$$\overline{\qquad\qquad\qquad}$$
$$o=\overline{pp\div y^2}-a+yy.$$

Hence $pp-ay^2+y^4=o$; and putting $\frac{RR}{4}=\frac{aa}{4}-pp$,

$y=\sqrt{\frac{a\pm R}{2}}$ (by Chap. III. Sect. III. Not. II.) and

$x=\sqrt{a-yy}=\sqrt{a\frac{-a\mp R}{2}}=\sqrt{\frac{a\mp R}{2}}$. Therefore

$x=\sqrt{\frac{a+R}{2}}$ for the greater, and $y=\sqrt{\frac{a-R}{2}}$ for the less number.

Synthesis. $\sqrt{\frac{a+R}{2}}\times\sqrt{\frac{a-R}{2}}=\sqrt{\frac{a^2-R^2}{4}}$

$$=\sqrt{\frac{a^2}{4}-\frac{a^2}{4}+pp}=p.$$

Again $\frac{a+R}{2}+\frac{a-R}{2}=\frac{2a}{2}=a$.

PROBL. XII. Given s and d, quære p or xy.

$$\begin{array}{ll} x+y=s & \text{Hence}\ \left.\begin{array}{l} x=\frac{s+d}{2} \\ y=\frac{s-d}{2}\end{array}\right\}\text{by Probl. I.} \\ x-y=d & \end{array}$$

$$\overline{\qquad\qquad}\times$$

Therefore $xy=\frac{ss-dd}{4}$.

Note. Here the solution itself contains the synthetical demonstration of the problem.

PROBL.

PROBL. XIII. If A and B together can perform a piece of work in time of a, A and C together in the time b, and B and C together in time c, in what time will each of them perform it alone?

Let A perform the work in the time x, B in y, and C in z; then as the rk is the same in all cases, it may be represented by unity.

the problem.	1	$x : 1 :: a : a \div x$	the work performed by A in a days.
	2	$y : 1 :: a : a \div y$	the work performed by B in a days.
	3	$x : 1 :: b : b \div x$	the work performed by A in b days.
	4	$z : 1 :: b : b \div z$	the work performed by C in b days,
	5	$y : 1 :: c : c \div y$	the work performed by B in c days.
	6	$z : 1 :: c : c \div y$	the work performed by C in c days.
	7	$\frac{a}{x} + \frac{a}{y} = 1$	
	8	$\frac{b}{x} + \frac{b}{z} = 1$	
	9	$\frac{c}{y} + \frac{c}{z} = 1$	
7th $\times bc$	10	$\frac{abc}{x} + \frac{abc}{y} = bc$	
8th $\times ac$	11	$\frac{abc}{x} + \frac{abc}{z} = ac$	
9th $\times ab$	12	$\frac{abc}{y} + \frac{abc}{z} = ab$	+
	13	$\frac{2abc}{x} + \frac{2abc}{y} + \frac{2abc}{z} = bc + ac + ab$	
13th $- 2 \times$ 10th	14	$\frac{2abc}{z} = ac + ab - bc$ and $z = \frac{2abc}{ac + ab - bc}$	
13th $- 2 \times$ 11th		$\frac{2abc}{y} = bc + ab - ac$ and $y = \frac{2abc}{bc + ab - ac}$	
13th $- 2 \times$ 12th		$\frac{2abc}{x} = bc + ac - ab$ and $x = \frac{2abc}{bc + ac - ab}$.	

Synthesis.

$$\left.\begin{array}{l}\dfrac{2abc}{bc+ac-ab} : 1 :: a : \dfrac{bc+ac-ab}{2bc} \\ \dfrac{2abc}{bc+ab-ac} : 1 :: a : \dfrac{bc+ab-ac}{2bc}\end{array}\right\}\text{the work of}\left\{\begin{array}{l}\text{A in } a \text{ days.}\\ \text{B in } a \text{ days.}\end{array}\right.$$

$$+$$

$$\frac{2bc}{2bc}=1, \text{ the work of A and B in } a \text{ days.}$$

$$\left.\begin{array}{l}\dfrac{2abc}{bc+ac-ab} : 1 :: b : \dfrac{bc+ac-ab}{2ac} \\ \dfrac{2abc}{ac+ab-bc} : 1 :: b : \dfrac{ac+ab-bc}{2ac}\end{array}\right\}\text{the work of}\left\{\begin{array}{l}\text{A in } b \text{ days.}\\ \text{C in } b \text{ days.}\end{array}\right.$$

$$+$$

$$\frac{2ac}{2ac}=1, \text{ the work of A and C in } b \text{ days}$$

$$\left.\begin{array}{l}\dfrac{2abc}{bc+ab-ac} : 1 :: c : \dfrac{bc+ab-ac}{2ab} \\ \dfrac{2abc}{ac+ab-bc} : 1 :: c : \dfrac{ac+ab-bc}{2ab}\end{array}\right\}\text{the work of}\left\{\begin{array}{l}\text{B in } c \text{ days.}\\ \text{C in } c \text{ days.}\end{array}\right.$$

$$+$$

$$\frac{2ab}{2ab}=1, \text{ the work of B and C in } c \text{ days.}$$

Note. It appears that a, b, c, muſt be ſuch, that the product of any two of them muſt be leſs than the ſum of theſe two multiplied by the third. This is neceſſary to give poſitive values of x, y, and z, which alone can take place in this queſtion. Beſides, if x, y, and z, be aſſumed as any known numbers whatever, and if values of a, b, and c, be deduced from the 7th, 8th, and 9th, ſteps of the preceding operation, it will appear, that a, b, and c, will have the property required in the limitation here mentioned.

If a, b, and c, were ſuch, that any of the quantities x, y, or z, became equal to 0, it implies, that one of the agents did nothing in the work. If the values of any of theſe quantities be negative the only ſuppoſition which could give them any meaning, would be, that ſome of the agents, inſtead of promoting the work, either obſtructed it, or undid it to a certain extent.

PROBL.

PROBL. XIV. A courier ſets out from a certain place, and travels at the rate of m miles in n hours; and a hours after, another ſets out from the ſame place, and travels the ſame road, at the rate of p miles in q hours: I demand how long and how far the firſt muſt travel, before he is overtaken by the ſecond?

Let the number of hours which the 1ſt travelled be	1	x
Then the 2d travelled	2	$x-a$
The 1ſt travelled $mx \div n$ miles in x hours, becauſe	3	$n : m :: x : mx \div n$
The 2d travelled $\frac{px-ap}{q}$ miles in $x-a$ hours, for	4	$q : p :: x-a : \frac{px-ap}{q}$
Therefore by queſtion	5	$\frac{px-ap}{q}=\frac{mx}{n}$ $\times nq$
By Multiplication	6	$npx-anp=mqx$
By Tranſpoſition	7	$npx-mqx=anp$
By Diviſion	8	$x=\frac{anp}{np-mq}$.
	9	$x-a=\frac{anp}{np-mq}-a=\frac{amq}{np-mq}$.

Syntheſis.

Synthesis.

$n : m :: \frac{anp}{np-mq} : \frac{amp}{np-mq}$ the travel of 1st courier.

$q : p :: \frac{amq}{np-mq} : \frac{amp}{np-mq}$ the travel of 2d courier.

Note. Here it is obvious, that np must be greater than mq, else the problem is impossible; for then the value of x would either be infinite (if $np=mq$ or $np-mq=0$) or negative (if $np<mq$ or $np-mq<0$). This limitation appears also from the nature of the problem, as the 2d courier must travel at a greater rate than the 1st, in order to overtake him. For the rate of the 1st courier is to the rate of the 2d as $m \div n$ to $p \div q$, that is, as mq to np; and therefore np must be greater than mq.

SECTION II.

DEMONSTRATION OF THEOREMS.

I. *Of Arithmetical Series.*

PROP. In an arithmetical series, the sum of the 1st and last terms is equal to the sum of any two intermediate terms, equally distant from the extremes.

Let the 1st term be a, the last x, and d the common difference; then $a+d$ will be the 2d, $x-b$ the last but one, &c.

Thus,

Thus, $a,\ a+d,\ a+2d,\ a+3d,\ a+4d,$ &c.
$x,\ x-d,\ x-2d,\ x-3d,\ x-4d,$ &c.

It is plain, that the terms in the same perpendicular rank are equally distant from the extremes, and that the sum of any two in it is $a+x$, the sum of the first and last.

COR. I. Hence the sum of all the terms of an arithmetical series, is equal to the sum of the first and last, taken half as often as there are terms. Therefore, if n be the number of terms, and s the sum of the series;

$s=\overline{a+x}\times\frac{n}{2}$. If $a=0$, $s=\frac{nx}{2}$.

Example.

What debt can be discharged in a year, by weekly payments in arithmetical progression, whereof the first term is 1 shilling, and the last is 5*l.* 3*s*.

Here we have $a=1$; $x=103$; $n=52$.

Therefore $s=\overline{1+103}\times\frac{52}{2}=104\times 26=2704s$.

The debt required is then 135*l.* 4*s*.

COR. II. The same notation being understood, since any term in the series consists of a, the first term, together with d taken as often as the number of terms preceding it, it follows, that

$$x=a+\overline{n-1}\times d=a+dn-d, \text{ and}$$
$$s=\overline{2a+dn-d}\times\frac{n}{2}=\frac{2an+dn^2-dn}{2}.$$

Therefore

Therefore, from the firſt term, the common difference, and number of terms being given, the ſum may be found.

Example.

A traveller wants to arrive at the end of his journey in 4 days, by haſtening his journey 3 leagues every day. To execute his deſign, he is obliged to go $20\frac{1}{2}$ leagues the firſt day. It is required to know the number of the leagues of his journey.

Here $a=20\frac{1}{2}$, $d=3$, $n=4$.

Then $s=\overline{41+12-3}\times 2=50\times 2=100$, the leagues required.

COR. III. Again, becauſe from the equation of Cor. I. we draw $an=2s-nx$, this value ſubſtituted in the 2d equation of Cor. II. gives $s=\frac{4s-2nx+dn^2-dn}{2}$, and then $s=\frac{2nx+dn-dn^2}{2}$. Hence from the laſt term, the common difference, and number of terms being given, the ſum may be found.

Example.

A man in 16 days went from London to a certain place, every days journey increaſing the former by 4, and the laſt he went 64 miles. How many miles is that place diſtant from London?

$n=16$, $d=4$, $x=64$.

$$s=\frac{2\times 16\times 64+4\times 16-4\times 16\times 16}{2}=544 \text{ mi.}$$

COR.

COR. IV. And since $n=\frac{x-a+d}{d}$ (Cor. II.) this value substituted in the equation $s=\overline{a+x}\times\frac{n}{2}$ (Cor. I.) gives $s=\overline{a+x}\times\frac{\overline{x-a+d}}{2d}$, or $s=\frac{x^2-a^2+ad+dx}{2d}$.
Whence the first and last terms, with the common difference, being given, the sum may be found.

Example.

Suppose 12 eggs were laid in a straight line, at one yard distance from each other; and the first egg, one yard distance, the last 12 yards, from a basket, into which a man is to put them one by one: how many yards must he walk?

Since the man must come back again, when he has walked one yard, the first term is 2, the common difference also 2, and therefore the last term must be 24 (Corollary II.) Hence $a^2=4$, $x^2=576$, and $s=\frac{576-4+4+48}{4}=\frac{624}{4}=156$ yards.

Cor. V. Hence any problem whatever, belonging to Arithmetical Series, may be resolved by one of the following five equations:

1st $s=\overline{a+x}\times\frac{n}{2}$;
2d $x=a+dn-d$;
3d $2s=2an+dn^2-dn$;
4th $2s=2nx+dn-dn^2$;
5th $2ds=x^2+dx-a^2+ad$.

 For

For of the four quantities of an equation, three being given, the fourth may be found, by resolving the equation in which they are contained; as may be seen by the following Examples.

Example I.

A man going from London to a certain place, distant 544 miles, went 4 miles the 1st day, and increased every day his journey in an arithmetical progression, so that the last day he went 64 miles: how many days was he upon his journey?

By equation 1st, $n=\dfrac{2s}{a+x}=\dfrac{1088}{68}=16$ days.

Example II.

A person being fined for a certain number of months, payed 6*l.* the first month, and 102*l.* the last; his fine increasing 12*l.* every month: how many months did he pay it?

By equation 2d, $n=\dfrac{x-a+d}{d}=\dfrac{102-6+12}{12}=9$ mo.

Example III.

In a heap of bullets disposed in an increasing arithmetical progression, I suppose 1st, that there were 18 rows, of which every row was greater by 2 bullets, than the row which preceded it; 2d, that there were 360 bullets in all: I demand how many bullets there were in the 1st row, and also how many there were in the last?

By

By equation 3d, $a=\frac{2s+dn-dn^2}{2n}=\frac{s}{n}+\frac{d}{2}-\frac{dn}{2}$

$=\frac{360}{68}+\frac{2}{2}-\frac{2.18}{2}=3$ bullets.

By equation 4th, $x=\frac{2s+dn^2-dn}{2n}=\frac{s}{n}+\frac{dn}{2}-\frac{d}{2}$

$=\frac{360}{18}+\frac{2.18}{2}-\frac{2}{2}=37$ bullets.

Example IV.

A man is to go to a certain place 544 miles diſtant from London, in 16 days, and he intends to go 4 miles the firſt day; by what common difference muſt he haſten every day his journey, in order to execute his deſign?

By equation 3d, $d=\frac{2s-2an}{n^2-n}$

$=\frac{1088-128}{256-16}=\frac{960}{240}=4$ miles.

Example V.

A gentleman buys an horſe for 82*l.* 8*s.* paying 5*s.* for his firſt nail, 8*s.* for the ſecond, 11*s.* for the third, and ſo on in an arithmetical progreſſion; I want to know the number of nails which were contained in this horſe's ſhoes.

By equation 3d, $n^2+\frac{2a-d}{d}\times n-\frac{2s}{d}=0$, that is,

$n^2+\frac{7n}{3}-\frac{3296}{3}=0.$

$$\text{Therefore } n = \frac{-7}{6} \pm \sqrt{\frac{3296}{3} + \frac{49}{36}}$$

$$= \frac{-7}{6} \pm \sqrt{\frac{39601}{36}}$$

$$= \frac{-7}{6} \pm \frac{199}{6} = 32 \text{ nails.}$$

It is plain, that 32 is the only root which anſwers to the queſtion; the other, $-34\frac{1}{3}$, being negative, has no ſignification.

II. *Of Geometrical Series.*

PROPOSITION I. THE product of the extremes in a geometrical ſeries, is equal to the product of any two terms equally diſtant from the extremes.

Let a be the firſt term, x the laſt, r the common ratio; then the ſeries is,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \&c.$$

$$x,\ \frac{x}{r},\ \frac{x}{r^2},\ \frac{x}{r^3},\ \frac{x}{r^4},\ \&c.$$

It is obvious, that any term in the upper rank is equally diſtant from the beginning, as that below it is from the end; and the product of any two ſuch is equal to ax, the product of the firſt and laſt.

PROP. II. The ſum of a geometrical ſeries, wanting the firſt term, is equal to the ſum of all but the laſt term, multiplied by the common ratio.

For,

For, assuming the preceding notation of a series, it is plain, that

$$ar+ar^2+ar^3, \&c. \ldots +\frac{x}{r^3}+\frac{x}{r^2}+\frac{x}{r}+x$$

$$=ra+ar+ar^2, \&c. \ldots +\frac{x}{r^4}+\frac{x}{r^3}+\frac{x}{r^2}+\frac{x}{r}.$$

COR. I. Therefore s being the sum of the series,

$$\overline{s-n}\times r=s-a; \text{ and } s=\frac{rx-a}{r-1}.$$

COR. II. Since the exponent of r in any term is equal to the number of terms preceding it; hence in the last term its exponent will be $n-1$; the last term therefore is

$$x=ar^{n-1}, \text{ and } s=\frac{ar^n-a}{r-1}=a\times\frac{r^n-1}{r-1}.$$

COR. III. If the series decreases, and the number of terms is infinite, then according to this notation, x, the least term, will be 0, and

$$s=\frac{-a}{r-1}=\frac{a}{1-r}, \text{ a finite series.}$$

Example.

Required the sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. to infinity.

Here $a=1$, and $r=\frac{1}{2}$. Therefore

$$s=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2.$$

Note.

Note. What are called in Arithmetic repeating and circulating decimals, are truly geometrical decreasing series, and therefore may be summed by this Rule.

Thus, .333, &c. $=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ &c. is a geometrical series, in which $a=\frac{3}{10}$, and $r=\frac{1}{10}$; therefore $s=\frac{3}{10}\div(1-\frac{1}{10})=\frac{3}{10}\div\frac{9}{10}=\frac{1}{3}$.

Thus also, .2424, &c. $=\frac{8}{33}$, for here $x=\frac{24}{100}$ and $r=\frac{1}{100}$; therefore $s=\frac{24}{100}\div(1-\frac{1}{100})=\frac{24}{100}\div\frac{99}{100}=\frac{8}{33}$.

COR. IV. Because $a=\frac{x}{r^{n-1}}$ (Cor II.) substituting this value in the equation $s=a\times\frac{r^n-1}{r-1}$, we find

$$s=\frac{x}{r^{n-1}}\times\frac{r^n-1}{r-1}.$$

Cor. V. Since $r=\frac{s-a}{s-x}$ (Cor. I.) and $r=\frac{\sqrt[n-1]{x}}{\sqrt[n-1]{a}}$ (Cor. II.) we get $\frac{s-a}{s-x}=\frac{\sqrt[n-1]{x}}{\sqrt[n-1]{a}}$ or $\overline{s-a}\times\sqrt[n-1]{a}$ $=\overline{s-x}\times\sqrt[n-1]{x}$; which equation, raised to the power $n-1$, becomes $a\times(s-a)^{n-1}=x\times(s-x)^{n-1}$.

COR.

COR. VI. Hence any problem whatever, upon geometrical ſeries, may be reſolved by one of the following five equations,

1ſt, $rs - rx = s - a$;

2d, $x = ar^{n-1}$;

3d, $rs - s = ar^n - a$;

4th, $sr^n - sr^{n-1} = xr^n - x$;

5th, $a \times (s-a)^{n-1} = x \times (s-x)^{n-1}$.

For of the four quantities of an equation, any three being given, the fourth may be found by reſolving the equation in which they are contained, as may be ſeen by the following Examples.

Example I.

By a lawyer's being obſtinate, it coſt a man in law-ſuits 121000*l*. The firſt law-ſuit coſt him 1000*l*. and the laſt 81000*l*. The expences of the others were mean proportionals between theſe two extremes; it is required to know in what proportion the law-ſuits have been increaſed; and how many they have been.

By equation 1ſt, $r = \frac{s-a}{s-x} = \frac{121000 - 1000}{121000 - 81000}$

$$= \frac{120000}{40000} = 3.$$

Again by equation 2d, $r^{n-1} = \frac{x}{a}$, that is,

$3^{n-1} = \frac{81000}{1000} = 81 = 3^4$. Therefore

$n - 1 = 4$ and $n = 5$ the number of the law-ſuits.

Example

Example II.

A person playing against another double or quits, loſt 10 times running. He played for 3*l.* the firſt time, what did he loſe the 10th ?

By equation 2d, $x = ar^{n-1} = 3 \times 2^{10-1} = 3 \times 2^{9}$
$= 1536.$

Example III.

We ſuppoſe that the population of a country has increaſed every year by $\frac{1}{10}$; how many ſouls were there at the beginning, if they are 14641 at the end of 5 years ?

By equation 2d, $a = \frac{x}{r^{n-1}} = \frac{14641}{\left(\frac{11}{10}\right)^4} = \frac{14641}{\frac{14641}{10000}} = 10000.$

Example IV.

A man skilled in numbers met with another, who expreſſing a great deſire of buying his coat, he agreed to let him have it, if he would give him only the price of the 16 buttons, reckoning 1*d.* for the 1ſt button, 2*d.* for the 2d, 4*d.* for the 3d, and the reſt thus, in double proportion: for how much did he ſell his coat ?

By equation 3d, $s = \frac{ar^{n} - a}{r - 1} = \frac{1.2^{16} - 1}{2 - 1} = 32767d.$

$= 136l.\ 10s.\ 7d.$

Example

Example V.

A rake ſpent all his fortune in 5 months; he quadrupled his expences every month, and the firſt month he ſpent 100*l.* what was his fortune?

By equation 3d, $s=\frac{ar^n-a}{r-1}=\frac{100.4^5-100}{5-1}$

$=\frac{102400-100}{4}=25575l.$

Example VI.

A perſon draws at five different times, from a caſk of wine, according to an increaſing geometrical progreſſion, whoſe laſt term is 243 pints, and whoſe common ratio is 3. How many pints were drawn the 1ſt time, and how many in all the five different times?

By equation 2d, $a=\frac{x}{r^n-1}=\frac{243}{3^4}=\frac{243}{81}=3.$

By equation 4th, $s=\frac{xr^n-x}{r^n-r^{n-1}}=\frac{243.3^5-243}{3^5-3^4}$

$=\frac{243.243-243}{243-81}=363.$

Note. I do not give any example for the equation 5th, becauſe the calculating becomes ſo complicate; and therefore I will treat this matter in the following Chap. V. Sect. I. where, by the uſe of logarithms, all the difficulty will be taken away.

III. *Of Algebraical Series.*

DEFINITION I. The *general term* of an algebraical series is an expression containing some powers of n (the number of terms); and some numbers, so that substituting 1, 2, 3, 4, &c. instead of n, the results give the 1st, 2d, 3d, 4th, &c. terms of the series itself.

Thus, $2n-1$ is the general term of the series

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \&c.$$

For taking $n=1$, we find $2n-1=1$.
$n=2$, $2n-1=3$.
$n=3$, &c. $2n-1=5$, &c.

DEFINITION II. The *general sum* of an algebraical series is an algebraic expression containing some powers of n; and some numbers, so that substituting 1, 2, 3, 4, &c. instead of n, the results give the value of 1, 2, 3, 4, &c. terms of the series itself.

Thus, n^2 is the general sum of the series

$$1,\ 3,\ 5,\ 7,\ 9,\ \&c.$$

For putting $n=1$, we find $n^2=1$.
$n=2$, $n^2=4$.
$n=3$, &c. $n^2=9$, &c.

Note I. The highest power of n in a general term of an algebraic series, is always equal to the order of the series itself; but in its general sum it surpasses that order by unity.

Note II. A general expression of the last term of an algebraical series, may be considered as its general term.

PROP.

PROP. I. To find the general term of a given algebraic series, take, according to the preceding Note I. some terms of the following indefinite series:

$$a+bn+cn^2+dn^3+en^4+fn^5, \text{ \&c.}$$

Then put successively, instead of n, the numbers 1, 2, 3, 4, &c. in order to have as many equations, as there are indeterminate coefficients a, b, c, d, &c. and, resolving these equations, you will find the value of those coefficients, which being substituted in the terms you have taken, will give the general term required.

This proposition is demonstrated by the progress itself.

Example.

Required the general term of the algebraic series of the 2d order.

$$1, 5, 12, 22, 35, 51, 70, 92, \text{ \&c.}$$

I take $a+bn+cn^2$, and putting $n=1, 2, 3,$
I have $a+b+c=1,$
$a+2b+4c=5,$
$a+3b+9c=12.$
Hence $b+3c=4$; $2c=3$; $c=\frac{3}{2}$;
$b+5c=7$; $b=-\frac{1}{2}$; $a=0$.

Therefore $\frac{3n^2}{2}-\frac{n}{2}$ is the general term required.

COR. An algebraic series being given, make another, so that its 1st, 2d, 3d, 4th, &c. terms be the value of the 1st, 2d, 3d, and 4th, &c. terms of the given series; find the general term of this new series by the proposition, and it will be the general sum of the series proposed.

Example.

Required the general ſum of the ſame.

Series, 1, 5, 12, 22, 35, 51, 70, 92, &c.
Its ſum, 1, 6, 18, 40, 75, 126, 196, 288, &c.

This new ſeries being of the third order, I take

The indefinite general term $a+bn+cn^2+dn^3$.

Therefore, by Prop. $a+b+c+d=1$,
$a+2b+4c+8d=6$,
$a+3b+9c+27d=18$,
$a+4b+16c+64d=40$.

Whence $b+3c+7d=5$; $2c+12d=7$; $6d=3$.
$b+5c+19d=12$; $2c+18d=10$.
$b+7c+37d=22$.

Therefore $d=\frac{1}{2}$, $c=\frac{1}{2}$, $b=0$, $a=0$.

And $\frac{n^3}{2}+\frac{n^2}{2}$, the general term of the ſeries 1, 6, 18, &c. as well as the general ſum of the propoſed ſeries 1, 5, 12, &c.

PROP. II. In a given general ſum ſubſtitute, inſtead of n, the quantity $n-1$, and ſubtract the reſult from that ſum, the remainder will be the general term anſwering to the given general ſum.

Let T be the general term, S the general ſum of terms n, and s the general ſum of terms $n-1$, it is plain that $S-s$ will be the laſt term, becauſe, if, for inſtance, from the ſum of five terms the ſum of four be ſubtracted, the remainder will be the value of the 5th, or laſt term. Therefore, $S-s$ being a general expreſſion of the laſt term, $S-s$ will be the general term (Def. II. Not. II.), that is, $T=S-s$.

Example.

Example.

Let $S=\frac{n^3}{2}+\frac{n^2}{2}$; then $s=\frac{(n-1)^3}{2}+\frac{(n-1)^2}{2}$,

or $s=\frac{n^3}{2}-\frac{2n^2}{2}+\frac{n}{2}$. Therefore

$T=S-s=* \quad \frac{3n^2}{2}-\frac{n}{2}$ the general term required.

COR. Substituting successively, in the general term, instead of n, the numbers 1, 2, 3, &c. the results will give the series belonging to the sum S.

Thus, in our case, the series is

1, 5, 12, 22, 35, 51, 70, 92, &c.

Because, if $n=1$, $\quad \frac{3n^2-n}{2}=1$.

$=2, \quad\quad =5$.

$=3$, &c, $\quad =12$, &c.

PROP. III. Raise the quantity $n-1$ to the power, by unity, greater than the highest power of n in a given general term; then multiplying that power by a fraction, whose numerator is the coefficient of this highest power, and the denominator its index increased by unity, add the product to the general term itself; and cutting off from the result the highest power of n, this power, if there is no remainder left, will be the general sum. But, if there is any remainder, this power will be only the 1st part of the general sum; then, considering the remainder as a general term, repeat the same operations as before, and so on, till there is no remainder; all the powers of n so found, and taken together, will give the general sum of the given general term.

For

For being $S-s=T$, it follows that $S=T+s$: then s is found by the method explained in the propoſition.

Example I.

Let $T=2n-1$; then $s=\frac{2\times\overline{n-1}^{2}}{2}$,

or $s=n^{2}-2n+1$. Therefore

$$\overline{\qquad\qquad}+$$

$S=T\times s=n^{2}\quad *\quad *$ the general ſum required.

Example II.

Let $T=\frac{3n^{2}}{2}-\frac{n}{2}$; then $s=\frac{3\div 2}{3}\times\overline{n-1}^{3}$,

or $s=\frac{n^{3}}{2}-\frac{3n^{2}}{2}+\frac{3n}{2}-\frac{1}{2}$. Therefore

$$\overline{\qquad\qquad\qquad\qquad}+$$

1ſt part of $S=\left.\frac{n^{3}}{2}\right)\ *\ +n-\frac{1}{2}$. Again

$s=\frac{1}{2}\cdot\overline{n-1}^{2}=\frac{n^{2}}{2}-n+\frac{1}{2}$. Therefore

$$\overline{\qquad\qquad\qquad\qquad}+$$

2d part of $S=\left.\frac{n^{2}}{2}\right)\ *\quad *$. Therefore

$$\overline{\qquad\qquad}+$$

$S=\frac{n^{3}}{2}+\frac{n^{2}}{2}$ the general ſum required.

COR. I. The ſeries belonging to the general ſum, is found as in the Corollary of Prop. II.

COR.

COR. II. If $T = n$,

$$= \frac{n}{1} \times \frac{n+1}{2},$$

$$= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3},$$

$$= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4};$$

Then will S be $= \frac{n}{1} \times \frac{n+1}{2}$,

$$= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3},$$

$$= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4},$$

$$= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5},$$

and so on, as will appear by drawing from the value of S by the proposition, and then by resolving it into its factors, from which resolution the law of continuation is manifest.

IV. *Of Figurate Numbers.*

DEFINITION. That algebraic series which arises by adding together a rank of

Units, called fig. numb. of the 1st order	is called a series of fig. numb. of the	2d	order.
Figurate numbers of the 2d order		3d	
3d order		4th	
4th order		5th	
5th order		6th	
6th order		7th	

Therefore

Therefore the figurate numbers of the

$$\left.\begin{array}{l}\text{1st order}\\ \text{2d order}\\ \text{3d order}\\ \text{4th order}\\ \text{5th order}\end{array}\right\}\text{ are }\left\{\begin{array}{llllll}1, & 1, & 1, & 1, & 1, & \&c.\\ 1, & 2, & 3, & 4, & 5, & \&c.\\ 1, & 3, & 6, & 10, & 15, & \&c.\\ 1, & 4, & 10, & 20, & 35, & \&c.\\ 1, & 5, & 15, & 35, & 70, & \&c.\end{array}\right.$$

COR. Hence it is manifest that each figurate number is equal to the sum of the preceding series so far as that number; and therefore, to find a general expression for a figurate number of any order, is the same thing as to find the general sum of the figurate numbers of the preceding order.

PROP. The general sum of the following figurate numbers:

1, 1, 1, 1, 1, &c. is n.

1, 2, 3, 4, 5, &c. is $\frac{n}{1} \times \frac{n+1}{2}$.

1, 3, 6, 10, 15, &c. is $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$.

1, 4, 10, 20, 35, &c. is $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$.

1, 5, 15, 35, 70, &c. is $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$.

It is evident by inspection, that the general sum of the series 1, 1, 1, 1, 1, &c. is n; but n is the general term of the series 1, 2, 3, 4, 5, &c. (Def. Cor.); therefore $\frac{n}{1} \times \frac{n+1}{2}$ is its general sum (see Algebr. Series, Prop. III. Cor. II. Again $\frac{n}{1} \times \frac{n+1}{2}$ is the gene-

ral

ral term of the series 1, 3, 6, 10, 15, &c. and $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$ its general sum (ib.); and so on. Whence the general sum of the figurate numbers as in the proposition.

COR. Hence universally the general sum of figurate numbers of order m, is

$$\frac{n \times \overline{n+1} \times \overline{n+2} \times \overline{n+3} \times \ldots\ldots \times \overline{n+m-1}}{1 \times 2 \times 3 \times 4 \times \ldots\ldots \times m},$$

in which the quantity $\frac{n+m-1}{m}$ signifies the last factor; thus, for instance, if the figurate numbers are of the 5th order, then $m=5$, and $\frac{n+m-1}{m}=\frac{n+5-1}{5}=\frac{n+4}{5}$, the last factor.

V. *Of Combinations.*

DEFINITION I. When some quantities are taken two by two, three by three, &c. without any repetition of them, and without including the different changing of place, the results are called *Combinations*. Thus, *ab*, *ac*, *bc*, are three combinations of letters *a*, *b*, *c*, taken two by two; but *aa*, *bb*, *cc*, as repetitions of the same letter, are not to be considered as combinations, and *ba*, *ca*, *cb*, being different from *ab*, *ac*, *bc*, only by changing of place, are no new combinations.

DEFINITION II. The *exponent* of a combination is the number expressing how many quantities there are in that combination. Thus, 2 is the exponent of com-

binations ab, ac, bc, because it shews that there are in each combination two quantities.

PROP. If n be the number of quantities to be combined, N their number of combinations, and m their exponent, it will be universally

$$N = \frac{n \times \overline{n-1} \times \overline{n-2} \times \ldots\ldots \times \overline{n-m+1}}{1 \times 2 \times 3 \times \ldots\ldots \times m}.$$

1st, *Let the exponent m, or the number of quantities a, b, c, d, &c. in each combination, be supposed only two.*

If n, the number of quantities to be combined, be only 2, as a and b, it is evident there can be only one combination, ab; but if n be increased by 1, or the quantities to be combined are supposed to become 3, as a, b, c, then it is plain that the number of combinations will be increased by 2, the number of the preceding quantities a and b, since with each of those, the new added quantity c may be combined; and therefore the whole number of combinations, in this case, will be truly expressed by $1+2$, viz. $N=1+2$. Again, if n be increased by 1 more, or the whole number of quantities be 4, as a, b, c, d, then it will appear that the number of combinations must be increased by 3, since the number of the preceding quantities is 3, and therefore will here be truly $N=1+2+3$. And, by reasoning in the same manner, it will appear that the whole number of combinations of 2, in 5 quantities, will be $N=1+2+3+4$; in 6 quantities, $N=1+2+3+4+5$; and in 7, $N=1+2+3+4+5+6$; &c. Whence universally, the number N of combinations of n quantities, taken 2 by 2, is $N=1+2+3+4+\ldots\ldots n-1$, a series of figurate numbers, the general sum whereof is $S=\frac{n \times \overline{n-1}}{1 \times 2}$ by what is above demonstrated.

2d, *Let*

2d, *Let now the exponent m be ſuppoſed* $= 3$.

It is plain that in 3 quantities a, b, c, there can be only one combination; but if n be increaſed by 1, or the number of quantities be a, b, c, d, then will the number of combinations be increaſed by 3, the number of all the combinations of two in the preceding letters a, b, c; ſince with each two of thoſe, the new letter d may be combined; therefore the number of combinations in this caſe is $1+3$. Again, if n be ſuppoſed to be increaſed by 1 more, or the number of letters to become 5, as a, b, c, d, e, then the number of combinations will be increaſed by 6 more ($=1+2+3$), or by all the combinations of two, in the 4 preceding letters a, b, c, d; ſince (as before) with each two of thoſe, the new letter e may be combined. Hence the number of combinations of n quantities, taken three by three, appears to be $1+3+6+10$, &c. continued to $n-2$ terms; which, being a ſeries of figurate numbers of the 3d order, the value thereof, by what is above demonſtrated, will be truly expreſſed by $S=\dfrac{n\times\overline{n-1}\times\overline{n-2}}{1\times 2\times 3}$.

And univerſally, ſince it appears, that increaſing the number of letters by 1, always increaſes the number of combinations by the whole number of combinations of the next inferior order, in all the preceding letters (for this obvious reaſon, that to each of theſe laſt combinations the new letter may be joined), it is manifeſt, that the combinations of any order obſerve the ſame law, and are generated in the very ſame manner as figurate numbers; and therefore may be exhibited by the ſame general expreſſions; only, as there are 2, 3, 4, or 5, &c. quantities given, or as the value of n is 2, 3, 4, or 5, when the number of terms in the ſeries thus generated is only one; according as thoſe quantities are taken 2 by 2, 3 by 3, 4 by 4, &c. it is plain that the number of terms muſt be leſs by 1, 2, or 3, &c. reſpectively,

ſpectively, than n the number of quantities; and therefore, inſtead of n in the foreſaid general expreſſions we muſt ſubſtitute, $n-1$, $n-2$, $n-3$, &c. reſpectively, in order to have the true value in this caſe. Hence the number of combinations of 2 quantities in n quantities, will be $\frac{n\times\overline{n-1}}{1\times 2}$; of 3, $\frac{n\times\overline{n-1}\times\overline{n-2}}{1\times 2\times 3}$; of 4, $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{1\times 2\times 3\times 4}$. Whence, univerſally, the number of combinations in any number n of quantities taken 2 by 2, 3 by 3, &c. will be expreſſed by

$$N=\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\ldots\ldots\times\overline{n-m+1}}{1\times 2\times 3\times 4\times\ldots\ldots\times m}.$$

Note. From this laſt general expreſſion, ſhewing the combination which any number of quantities will admit of, the theorem laid down in the next pages (where of infinite ſeries) for raiſing a binomial to any given power n, is very eaſily and naturally derived, as may be ſeen in Simpſon's Treatiſe of Algebra, p. 199, &c.

COR. Becauſe a quantity as a has only 1 combination; 2 quantities, as a, b, have 3 combinations; viz. a, b, ab; 3 quantities, as a, b, c, have 7 combinations; viz. a, b, c, ab, ac, bc, abc, and ſo on, the number of theſe combinations will be expreſſed by the terms of the following ſeries:

$$1,\ 3,\ 7,\ 15,\ 31,\ 63,\ 127,\ 255,\ 511,\ \&c.$$

that is, the 1ſt term of this ſeries gives the number of all the combinations of one quantity, the 2d gives the number of all the combinations of 2 quantities, the 3d of 3, the 4th of 4, and in general, the term n gives the number of all the combinations of n quantities. But $1=2-1$, $3=4-1$, $7=8-1$, $15=16-1$, &c. There-

Therefore, the terms of our ſeries ariſe from the difference between the terms of the two following:

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \ldots\ldots,\ 2^n$$
$$1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ldots\ldots,\ 1,$$

Whence the general term of our ſeries is $2^n - 1$, which expreſſion will conſequently repreſent the number of all the combinations of n quantities.

SCHOLIUM. The problem of dies, that is, in how many ways a number n of dies may be combined, in relation to the numbers 1, 2, 3, 4, 5, 6, which are upon their faces, is to be reſolved in a different manner. If the dies are two, or $n=2$, the number of combinations is 36, or $N=6^2$; becauſe a number of one die being combined with the 6 numbers of the other, gives 6 different combinations, 2 numbers give $6+6=12$ combinations, 3 give $6+6+6=18$, and ſo on; conſequently ſix numbers give $6+6+6+6+6+6=36=6^2$ combinations. If the dies are 3, or $n=3$, the number of combinations, or N will be $=216=6\times6\times6=6^3$; becauſe one number of the 3d die, combined with the preceding 36 combinations of the 2d dies, will give 6 times 36 combinations, that is, $N=216=6\times6=6^3$. Hence, in general, the number of combinations of n dies will be $N=6^n$.

II. *Of Permutations.*

DEFINITION. When ſome quantities are diſpoſed in all poſſible different manners, the reſults are called *Permutations*. Thus, all the permutations of letters a, b, c, are the ſix following:

$$abc,\ acb,\ cab,\ bac,\ bca,\ cba;$$

Note;

Note. Permutation concerns only the order of quantities.

PROP. If n be the number of quantities to be permuted, and N the number of permutations, it will be, in general,

$$N = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots\ldots \times n.$$

If $n = 2$, or if the number of quantities to be permuted be only 2, as a, and b, the permutations are only ab, ba, and then $N = 1 \times 2$. But if n be increased by 1, or the quantities to be permuted are supposed to become 3, as a, b, c, then it is plain that the new quantity c may be either the 1st, 2d, or 3d, in order, in each of the two preceding permutations, and it will therefore give the six following permutations:

$$cab,\ acb,\ abc,\ cba,\ bca,\ bac.$$

Hence in this case, $N = 1 \times 2 \times 3$, that is the whole number of permutations will be three times the last preceding. And, by the same reasoning, it will appear that the whole number of permutations of 4, 5, 6, &c. quantities will be 4 times, 5 times, 6 times, &c. the last preceding, because 4, 5, 6, &c. quantities may obtain 4, 5, 6, &c. different places in each of the last preceding permutations. Therefore, in general,

$$N = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots\ldots \times n.$$

Example I.

How many changes may be rung on 7 bells?

Answ. $N = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$ changes.

Example

Example II.

In how many different positions may 8 persons sit at table?

Ans. $N = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320$ positions.

Example III.

A gentleman desirous to board in a genteel family, demanded what were the terms for a year. Being answered 100*l.* he cunningly asked what he must give for so long as himself and 8 other gentlemen of his friends could sit every day at dinner in a different order. The host, thinking it would not be long, told him 15*l.* to which they agreed. How long were they to board in the house?

Ans. $N = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 362880$ days.

VII. *Of Infinite Series.*

It was observed, p. 77, that in many cases, if the division and evolution of compound quantities be actually performed, the quotients and roots can only be expressed by a series of terms, which may be continued *in infinitum.* By comparing a few of the first terms, the law of the progression of such a series will frequently be discovered, by which it may be continued without any farther operation. When this cannot be done, the work is much facilitated by several methods; the chief of which is that by the *binomial theorem.*

Theorem. Any binomial, as $a+b$, may be raised to any power m by the following rules:

1. From

1. From inspecting a table of the powers of a binomial obtained by multiplication, it appears that the terms without their coefficients, are as follows:

$$a^m,\ a^{m-1}b,\ a^{m-2}b^2,\ a^{m-3}b^3,\ \&c.$$

2. The coefficients of these terms will be found by the following Rule:

Divide the exponent of a in any term by the exponent of b increased by 1, and the quotient multiplied by the coefficient of that term, will give the coefficient of the next following term.

This Rule is found, upon trial in the table of powers, to hold universally. The coefficient of the 1st term is always 1, and, by applying the general Rule now proposed, the coefficients of the terms in order will be as follows:

$$1,\ m,\ m\times\frac{m-1}{2},\ m\times\frac{m-1}{2}\times\frac{m-2}{3},\ \&c.$$

They may be more conveniently expressed, thus,

$$1,\ Am,\ B\times\frac{m-1}{2},\ C\times\frac{m-2}{3},\ D\times\frac{m-3}{4},\ \&c.$$

the capitals denoting the preceding coefficient. Hence

$$(a+b)^m = a^m + ma^{m-1}b + m\times\frac{m-1}{2}\times a^{m-2}b^2$$

$$+ m\times\frac{m-1}{2}\times\frac{m-2}{3}\times a^{m-3}b^3 +,\ \&c.$$

or $(a+b)^m = a^m + Ama^{m-1}b + B\times\frac{m-1}{2}\times a^{m-2}b^2$,

$$+ C\times\frac{m-2}{3}\times a^{m-3}b^3 +,\ \&c.$$

Note.

Note. This is the celebrated *binomial theorem*: it is deduced here by induction only, but it may be rigidly demonstrated, though upon principles which do not belong to this place. (See the Note to the Proposition upon Combinations.)

COR. I. As m may denote any number integral or fractional, positive or negative; hence the evolution and division, as well as the involution, of a binomial, may be performed by this theorem.

Example I.

Let $m=\frac{1}{2}$, then the square root of $a+b$ will be as follows:

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\tfrac{1}{2}a^{\frac{-1}{2}}b+\tfrac{1}{2}\times\frac{-1}{4}\times a^{\frac{-3}{2}}b^{2}+\&c.$$

$$\text{or } (a+b)^{\frac{1}{2}}=\sqrt{a}+\frac{b}{2\sqrt{a}}-\frac{b^{2}}{8a\sqrt{a}}+\frac{b^{3}}{16a^{2}\sqrt{a}},\ \&c.$$

This being applied to the extraction of the square root of a^2+x^2 (by inserting a^2 for a, and x^2 for b) the same series result as formerly, that is,

$$\text{or } (a^{2}+x^{2})^{\frac{1}{2}}=a+\frac{x^{2}}{2a}-\frac{x^{4}}{8a^{3}}+\frac{x^{6}}{16a^{5}},\ \&c.$$

Example II.

If $\frac{1}{1-r}$ is to be turned into an infinite series, since

$$\frac{1}{1-r}=1\times(1-r)^{-1}, \text{ let } a=1, \ b=-r,$$

and $m=-1$; and the same series will arise as was obtained by division (   ), that is,

$$(1-r)^{-1}=1+r+r^{2}+r^{3}+r^{4}+\&c.$$

 Example

Example III.

Let $\frac{r^2}{\sqrt{2rz-zz}}$ be expressed by an infinite series.

Since $\frac{r^2}{\sqrt{2rz-zz}} = r^2 \times (2rz-zz)^{\frac{-1}{2}}$, by supposing $a=2rz$, $b=-zz$, and $m=-\frac{1}{2}$, then multiplying that series by r^2, and reducing the product, we find

$$r^2 \times (2rz-z^2)^{\frac{-1}{2}} = \frac{r^2}{\sqrt{2rz}} + \frac{z}{4\sqrt{2rz}} + \frac{3z^2}{32\sqrt{2rz}} + \&c.$$

COR. II. If the quantity expressed by b is negative, or if the binomial is $a-b$, then the even terms of the general power are all negative; and therefore it is

$$(a-b)^m = a^m - ma^{m-1}b + m \times \frac{m-1}{2} \times a^{m-2}b^2$$

$$- m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}b^3, \ \&c.$$

SCHOLIUM. This theorem may be applied to quantities which consist of more than two parts, by supposing them distinguished into two, and then substituting for the powers of these compound parts, their values, to be obtained also, if required, from the theorem. Thus,

$(c+d+e)^2 = (a+b)^2$, supposing $d+e=b$;
$(c+d+e+f)^2 = a+b^2$, supposing $d+e+f=b$,
or also supposing $c+d=a$, and $e+f=b$.

VIII. *Pro-*

VIII. *Properties of Numbers.*

THEOREM I. If a and b be any two quantities, of which the sum may be denoted by s, the difference by d, and their product by p, then the following propositions will be true.

1. $a^2+b^2=s^2-2p$,
2. $a^2-b^2=sd$,
3. $a^3+b^3=s^3-3ps$,
4. $a^3-b^3=s^2d-dp$,
5. $a^4+b^4=s^4-4ps^2+2p^2$,
6. $a^4-b^4=s^3d-2sdp$, &c.

Note. It is unnecessary to express these propositions in words, and the demonstrations are very easy, either by raising $a+b$ to certain powers, and making proper substitutions, or substituting the values of s, d, and p. Thus,

By the 1st method.

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3=s^3$$
$$=a^3+3ab\times\overline{a+b}+b^3=s^3$$
$$=a^3+3ps+b^3 \qquad =s^3$$

Hence $a^3+b^3=s^3-3ps$.

By the 2d method.

$$s^2d-dp=(s^2-p)\times d=(\overline{a+b}^2-ab)\times(a-b)$$
$$=(a^2+ab+b^2)\times(a-b)=a^3-b^3.$$

 LEMMA

LEMMA I. Let r be any number, and n any integer, $r^n - 1$ is divisible by $r - 1$.

The quotient will be $r^{n-1} + r^{n-2}$, &c. till the index of r be 0, and then the last term of it will be 1; for if this series be multiplied by $r - 1$, the divisor, it will produce the dividend $r^n - 1$. It will appear also by performing the division, and inserting for n any number.

LEMMA II. Let r be any number, and n any integer odd number, $r^n + 1$ is divisible by $r + 1$. Also if n be any even number, $r^n - 1$ is divisible by $r + 1$.

The quotient in both cases is $r^{n-1} - r^{n-2} + r^{n-3}$, &c. till the exponent of r be 0, and the last term $r^0 = 1$. If this series consist of an odd number of terms, and be multiplied by $r + 1$, the divisor, the product is $r^n + 1$, the dividend. If the series consist of an even number of terms, the product is $r^n - 1$. But it is plain, that the number of terms will be odd only when n is even. The conclusion will be manifest by performing the division.

LEMMA III. If r is the root of an arithmetical scale, and a, b, c, &c. the coefficients or digits, any number in that scale may be represented in the following manner, $a + br + cr^2 + dr^3 + er^4$, &c.

This proposition is evident. In the common scale $r = 10$, and the number 256, for instance, may be expressed thus,

$$6 + 5 \times 10 + 2 \times 10^2 = 6 + 50 + 200 = 256.$$

See Principle VI. p. 24.

THEOREM II. If from any number in the general scale now described, the sum of its digits be subtracted, the remainder is divisible by $r - 1$.

The

The number is $a+br+cr^2+dr^3+er^4$, &c. and the sum of the digits $a+b+c+d+e$, &c. (by Lemma III.)

Subtracting the latter from the former, the remainder is

$$br-b+cr^2-c+dr^3-d+er^4-e, \text{ \&c.}$$
$$=b\times\overline{r-1}+c\times\overline{r^2-1}+d\times\overline{r^3-1}+e\times\overline{r^4-e}, \text{ \&c.}$$

But by Lemma I. r^n-1 is divisible by $r-1$, whatever the integer number, n, may be, and therefore any multiple of r^n-1 is also divisible by $r-1$. Hence each of the terms $b\times\overline{r-1}$, $c\times\overline{r^2-1}$, &c. is divisible by $r-1$, and therefore the whole is divisible by $r-1$.

COR. I. Any number, the sum of whose digits is divisible by $r-1$, is itself divisible by $r-1$.

Let the number be called N, and the sum of the digits S; then by this Proposition, N$-$S is divisible by $r-1$; but S is supposed to be divisible by $r-1$; therefore it is plain that N must also be divisible by $r-1$.

COR. II. Any number, the sum of whose digits is divisible by an aliquot part of $r-1$, is also divisible by that aliquot part. For, since N$-$S (Theor. II.) is divisible by $r-1$, it is also divisible by an aliquot part of $r-1$; but S is supposed to be divisible by that aliquot part; therefore N is also divisible by the same aliquot part.

COR. III. This Theorem, with the Corollaries, relates to any scale whatever. It includes therefore the well-known property of 9 and of 3 its aliquot part, in the decimal or common scale; for, since $r=10$, $r-1=9$.

THEOREM III. In any number, if from the sum of the coefficients of the odd powers of r the sum of the coefficients of the even powers be subtracted, and the remainder added to the number itself, the sum will be divisible by $r+1$.

In

In the number $a+br+cr^2+dr^3+er^4+fr^5$, &c. the sum of the coefficients of the even powers of r is $b+d+f$, &c. the sum of the coefficients of the even powers of r is $a+c+e$, &c. If the latter sum be subtracted from the former, and the remainder added to the given number, it makes

$$br+b+cr^2-c+dr^3+d+er^4-e+fr^5+f, \text{ \&c.}$$
$$=b.\overline{r+1}+c.\overline{r^2-1}+d.\overline{r^3+1}+e.\overline{r^4-1}+f.\overline{r^5+1}, \text{ \&c.}$$

But (by Lemma II.) $r+1$, r^2-1, r^3+1, r^4-1, r^5+1, &c. are each divisible by $r+1$, and therefore any multiples of them are also divisible by $r+1$, hence the whole number is divisible by $r+1$.

COR. I. If the difference of the sum of the even digits, and the sum of the odd digits of any number be divisible by $r+1$, the number itself is divisible by $r+1$.

Let the sum of the even digits (that is, the coefficients of the odd powers of r) be S, the sum of the odd digits (that is, the coefficients of the even powers of r) be s, and let the number be N. Then by the Theorem $N+S-s$ is divisible by $r+1$, and it is supposed that $S-s$ is divisible by $r+1$; therefore N is divisible by $r+1$.

COR. II. In like manner, if $S-s$ is divisible by an aliquot part of $r+1$, N will be divisible by that aliquot part.

COR. III. If a number wants all the odd powers of r, or if it wants all the even powers of r, and if the sum of its digits be divisible by $r+1$, that number is divisible by $r+1$.

COR.

COR. IV. In the common ſcale $r+1=11$, which therefore will have the properties mentioned in this theorem, and the corollaries. Thus, in the number 64834, the ſum of the even digits is 7, the ſum of the odd digits is 18, and the difference is 11, a number diviſible by 11, the given number therefore (Cor. I.) is diviſible by 11. Thus alſo, the ſum of the digits of 7040308 is diviſible by 11, and therefore the number is diviſible by 11. (Cor. III.)

SCOLIUM. Theſe Theorems relate to any ſcale whatever, and therefore the properties of $r-1$ in Theorem II. would in a ſcale of *eight* belong to *ſeven*, and thoſe of $r+1$ in Theorem III. to *nine*. If *twelve* was the root of the ſcale, the former properties would belong to *eleven*, and the latter to *thirteen*.

CHAPTER V.

SOME APPLICATIONS OF ALGEBRA.

ALGEBRA has been ſucceſsfully applied to almoſt every branch of Mathematics; and the principles of theſe branches are often advantageouſly introduced into algebraical calculations.

In this place ſhall be given ſome examples of its application, 1ſt, to thoſe equations whoſe reſolution require the uſe of logarithms; 2d, to natural philoſophy or to phyſics; and 3d, to the practical calculations of buſineſs.

SEC-

SECTION I.

EXAMPLES OF THE USE OF LOGARITHMS, IN RESOLVING EQUATIONS.

PROPOSITION I. To resolve the equation $x=\frac{a^n b}{c^n}$.

By the properties of logarithms (Chap. II. Sect. IV.) the operation will be as follows:

$$\text{l}.x=\text{l}.\frac{a^n b}{c^n}=\text{l}.a^n+\text{l}.b-\text{l}.c^n=n\text{l}.a+\text{l}.b-n\text{l}.c.$$

If $n\text{l}.a+\text{l}.b-n\text{l}.c=\text{l.N}$, then $\text{l}.x=\text{l.N}$ and $x=\text{N}$.

Example.

Suppose there are 100000 inhabitants in a province, and that its population augments every year by a thirtieth part; what will be the number of its inhabitants at the end of an age?

Let x be the number of the inhabitants: then, because every 30 people become 31, at the end of every year, the population in our province will increase as follows:

$30:31::100000:x=100000(\frac{31}{30})$ at the end of 1st year.
$30:31::100000(\frac{31}{30})^1:x=100000(\frac{31}{30})^2$ - - - - - 2d year.
$30:31::100000(\frac{31}{30})^2:x=100000(\frac{31}{30})^3$ - - - - - 3d year.

And therefore the number of inhabitants at the end of 100 years will be $x=100000(\frac{31}{30})^{100}$. Hence by logarithms

l.x

$\mathrm{l}.x = \mathrm{l}.100000(\frac{31}{30})^{200} = \mathrm{l}.100000 + \mathrm{l}.(\frac{31}{30})^{200}$
$= \mathrm{l}.100000 + 100\mathrm{l}.(\frac{31}{30}) = \mathrm{l}.100000 + 100\mathrm{l}.31 - 100\mathrm{l}.30.$

But $\mathrm{l}.100000 = 5$ and $\mathrm{l}.31 - \mathrm{l}.30 = 0,014240439$.

Therefore $\mathrm{l}.x = 5 + 100 \times 0,014240439$
$= 5 + 1,424043900 = 6,4240439$, and $x = 2654874$, by Tables.

PROP. II. To resolve the equation $a\left(\frac{b+x}{x}\right)^n = c$.

$\mathrm{l}.a\left(\frac{b+x}{x}\right)^n = c$, or $\mathrm{l}.a + n\mathrm{l}.\frac{b+x}{x} = \mathrm{l}.c$,

that is $n\mathrm{l}.\frac{b+x}{x} = \mathrm{l}.c - \mathrm{l}.a$,

whence $\mathrm{l}.\frac{b+x}{x} = \frac{\mathrm{l}.c - \mathrm{l}.a}{n}$.

Suppose $\frac{\mathrm{l}.c - \mathrm{l}.a}{n} = \mathrm{l}.N$, then $\mathrm{l}.\frac{b+x}{x} = \mathrm{l}.N$,

$\frac{b+x}{x} = N$, and lastly $x = \frac{b}{N-1}$.

Example I.

The earth having been repeopled after the deluge by the three sons of Noah and their three wives; it is demanded in what proportion population must have increased every year, that there might be 1000000 men at the end of 200 years?

Let 1 be the annual increase for every x men, so that x men, at the end of every year, may be found $x+1$. Therefore in our case

$x : x+1 :: 6 : 6\left(\frac{x+1}{x}\right)^1$ for a year.

$$x : x+1 :: 6\left(\frac{x+1}{x}\right) : 6\left(\frac{x+1}{x}\right)^2 \text{ for 2 years.}$$

and consequently $6\left(\frac{x+1}{x}\right)^{200}$ for 200 years.

Hence $6\left(\frac{x+1}{x}\right)^{200} = 1000000$ by question, or

$l.6\left(\frac{x+1}{x}\right)^{200} = l.1000000$ by logarithms.

$$l.6 + 200 l.\left(\frac{x+1}{x}\right) = l.1000000,$$

$$l.\frac{x+1}{x} = \frac{l.1000000 - l.6}{200} = \frac{5,2218487}{200}$$

$$= 0,0261092435 = l.1,061963 \text{ by the Tables,}$$

Hence $l.\frac{x+1}{x} = l.1,061963$; $\frac{x+1}{x} = 1,061963$;

and $x = \frac{1}{0,061963} = 16$ men nearly.

Example II.

It is required to know the number, by which a people must increase every year, to be double at the end of every century.

The same notation of the increase being supposed, and taking 1 to represent the people, we shall find after a century.

$$1\left(\frac{x+1}{x}\right)^{100} = 2, \text{ or } l.\frac{x+1}{x} = \frac{l.2}{100}.$$

Hence by Tables, $l.\frac{x+1}{x} = \frac{0,3010300}{100}$

$$= 0,0030103 = l.1,0069555.$$

That

That is $l.\frac{x+1}{x}=l.1,0069555$; $\frac{x+1}{x}=1,0069555$;

and $x=\frac{1}{0,0069555}=144$ men nearly.

Note. From these calculations it appears, that the objections against the population of the old world oppose the evidence.

PROP. III. To resolve the equation $a^x=b$.

$l.a^x=l.b$, or $xl.a=l.b$, and $x=l.b\div l.a$.

Example.

Suppose that a certain number of men increase every year by the hundredth part, how long will they be before they are ten times as many as at first?

Let the number of men be represented by unity, then

$100 : 101 :: 1 : 1(\frac{101}{100})$ for a year.

$100 : 101 :: 1 : 1(\frac{101}{100})^2$ for 2 years.

And consequently $1(\frac{101}{100})^x$ for x years.

Hence $1(\frac{101}{100})^x=10$ by the question;

and $xl.\frac{101}{100}=l.10$ by logarithms.

Therefore $x(l.101-l.100)=l.10$;

$$x(0,0043214)=1=1,0000000; \text{ and}$$

$$x=\frac{1,0000000}{0,0043214}=231 \text{ years nearly.}$$

PROP. IV. To resolve the equation $a\times(s-a)^{n-1}=x\times(s-x)^{n-1}$.

$l.a+\overline{n-1}l.(s-a)=l.x+\overline{n-1}l.(s-x)$,

 Note;

Note. This logarithmical equation may be differently worked according to the unknown quantity required. I will consider the case, in which the value of the quantity s is to be required. Divide the said equation by $n-1$, and it will be

$$\frac{\mathrm{l}.a}{n-1}+\mathrm{l}.(s-a)=\frac{\mathrm{l}.x}{n-1}+\mathrm{l}.(s-x)\text{; then}$$

$$\mathrm{l}.(s-a)-\mathrm{l}.(s-x)=\frac{\mathrm{l}.x-\mathrm{l}.a}{n-1}\text{ by transposition.}$$

Suppose $\frac{\mathrm{l}.x-\mathrm{l}.a}{n-1}=\mathrm{l}.\mathrm{N}$, then

$$\mathrm{l}.(s-a)-\mathrm{l}.(s-x)=\mathrm{l}.\mathrm{N}\text{, or }\mathrm{l}.\frac{s-a}{s-x}=\mathrm{l}.\mathrm{N}.$$

Hence $\frac{s-a}{s-x}=\mathrm{N}$, and $s=\frac{\mathrm{N}x-a}{\mathrm{N}-1}$.

Example.

A man sold a house, in which were 12 rooms, on condition that the purchaser should pay only $3d.$ for the first room, and then for the others such a geometrical progression, that the last room should amount to $2214l.\ 6s.\ 2d.$ What did the house cost?

Here $a=3d.$ $x=531441d.$ $n=12.$ Therefore

$$\mathrm{l}.\mathrm{N}=\frac{\mathrm{l}.x-\mathrm{l}.a}{n-1}=\frac{\mathrm{l}.531441-\mathrm{l}.3}{11}=\frac{5,24832-0,47712}{11}$$
$$=0,47712=\mathrm{l}.3.$$

Whence $\mathrm{N}=3$, and consequently

$$s=\frac{\mathrm{N}x-a}{\mathrm{N}-1}=\frac{3\times531441-3}{3-1}=797160d.$$

The house therefore cost $3321l.\ 10s.$

SEC-

SECTION II.

EXAMPLES OF PHYSICAL PROBLEMS.

THE use of Algebra in Natural Philosophy may be properly illustrated by some examples of physical problems. The solution of such problems must be derived from known physical laws, which, though ultimately founded on experience, are here assumed as principles, and reasoned upon mathematically. The experiments by which the principles are ascertained admit of various degrees of accuracy; and on the degree of physical accuracy in the principles will depend the physical accuracy of the conclusions mathematically deduced from them. If the principles are inaccurate, the conclusions must in like manner be inaccurate; and, if the limits of inaccuracy in the principles can be ascertained, the corresponding limits, in the conclusions derived from them, may likewise be calculated.

PROBLEM I. We know, by the observations of Galileo, that the space passed through by a body falling from a state of rest, by the power of gravity, increases in the progression of the uneven numbers 1, 3, 5, 7, &c. that is to say, that a body, only by the force of gravity, passes through a space of 15 feet nearly in the 1st *second* of its fall, 45 in the 2d *second*, 75 in the 3d, and thus in progression. It is required to know how many feet it will have passed through in 6 *seconds*.

SO-

SOLUTION. The progreſſion 15, 45, 75, &c. is arithmetical, in which the 1ſt term $a=15$, the difference $d=30$, and the number of terms $n=6$. Therefore by the property of arithmetical ſeries the ſum

$$s=\frac{2an+dn^2-dn}{2}=540 \text{ feet.}$$

PROBL. II. Let a glaſs tube, a inches long, be filled with mercury, excepting b inches; and let it be inverted, as in the Torricellian experiment, ſo that the b inches of common air may riſe to the top: it is required to find at what height the mercury will remain ſuſpended, the mercury in the barometer being at that time c inches high.

Note. The ſolution of this problem depends upon the following principles:

1. The preſſure of the atmoſphere is meaſured by the column of mercury in the barometer; and the elaſtic force of the air, in its natural ſtate, which reſiſts the preſſure, is therefore meaſured by the ſame column.

2. In different ſtates, the elaſtic force of the air is reciprocally as the ſpaces which it occupies.

3. In this experiment the mercury that remains ſuſpended in the tube, together with the elaſtic force of the air in the top of it, being a counterbalance to the preſſure of the atmoſphere, may therefore be expreſſed by the column of mercury in the barometer.

SOLUTION. Let the mercury in the tube be x inches, the air in the top of it now occupies the ſpace $a-x$; it occupied formerly b inches, and its elaſtic force was c inches of mercury: now, therefore, the force muſt be $\frac{bc}{a-x}$ inches, for $\frac{1}{b}:\frac{1}{a-x}::c:\frac{bc}{a-x}$

(Princ.

(Princ. 2.) Therefore $x+\frac{bc}{a-x}=c$ (Princ. 3.) This reduced, and putting $a+c=2d$, and $b-a=e$, the equation is $x^2-2dx=ce$. This refolved gives $x=d\pm\sqrt{(ce+dd)}$.

APPLICATION. Let $a=30$, $b=8$, $c=28$; then $d=\frac{a+c}{2}=29$, and $e=b-a=-22$. Therefore $x=29\pm15$, that is $x=44$ or 14.

One of the roots 44 is plainly excluded in this cafe, and the other 14 is the true anfwer.

Note. If the column of mercury x, fufpended in the tube, were a counterbalance to the preffure of the atmofphere, expreffed by the height of the barometer c, together with the meafure of the elaftic force of b inches of common air, in the fpace of $x-a$; that is, if $x=c+\frac{bc}{x-a}$ or $x-\frac{bc}{x-a}=c$, the equation will be the fame as before, and the root 44 would be the true anfwer. But the experiment in this queftion does not admit of fuch a fuppofition.

PROBL. III. The diftance of the earth and moon d, and their quantities of matter e, m, being given, to find the point of equal attraction between them.

SOLUTION. Let the diftance of the point from the earth be x; its diftance from the moon will be therefore $d-x$. But gravitation is as the matter *directly*, and as the fquare of the diftance *inverfely*; therefore the earth's attraction is as $\frac{e}{x^2}$, and the moon's

attraction

attraction is as $\frac{m}{(d-x)^2}$. But these are here equal; therefore

$$\frac{e}{x^2}=\frac{m}{(d-x)^2} \text{ or } \frac{\sqrt{e}}{x}=\frac{\sqrt{m}}{d-x} \text{ by evolution.}$$

This equation reduced gives $x=\frac{d\sqrt{e}}{\sqrt{e}+\sqrt{m}}$

Or multiplying the numerator and the denominator by $\sqrt{e}-\sqrt{m}$ $x=\frac{de-d\sqrt{em}}{e-m}$.

APPLICATION. In round numbers, let $d=60$ semidiameters of the earth, $e=40$, $m=1$, then $x=52$ semidiameters nearly.

Note. There is another point beyond the moon at which the attractions are equal, and it would be found by resolving the first equation $\frac{e}{x^2}=\frac{m}{(d-x)^2}$ as follows:

$dde-2dex+ex^2=mx^2$ by clearing of fractions;

$(e-m)\times x^2-2dex=-dde$ by transposition;

$x^2-\frac{2dex}{e-m}=\frac{-dde}{e-m}$ by division;

$x=\frac{de\pm d\sqrt{em}}{e-m}$ by resolution.

Hence $x=\frac{de-d\sqrt{em}}{e-m}$, as before, and $x=\frac{de+d\sqrt{em}}{e-m}$; which expression in round numbers gives $x=71$ nearly.

PROBL.

PROBL. IV. Let a ſtone be dropped into an empty pit; and let the time of the dropping of it to the hearing the ſound from the bottom be given: to find the depth of the pit.

SOLUTION. Let the given time be a; let the fall of a heavy body in the 1ſt ſecond of time be b; alſo, let the motion of ſound in a ſecond be c.

Let the time of the ſtone's fall be	1	x
The time in which the ſound of it moves to the top is	2	$a-x$
The deſcent of a falling body is as the ſquare of the time; therefore being $1^2 : x^2 :: 6 : bx^2$, the depth of the pit is	3	bx^2
The depth from the motion of ſound (being $1 : c :: a-x : ca-cx$) is alſo	4	$ca-cx$
Therefore 3 and 4	5	$bx^2 = ca-cx$

This equation being reſolved, gives the value of x, and from it may be got bx^2 or $ca-cx$, the depth of the pit.

If the time is $a=10$ ſeconds, $b=15$ feet, and $c=1142$ feet, then $x=8.9$ nearly, and the depth is 1188 feet nearly.

Note. There are ſeveral circumſtances in this problem which render the concluſion inaccurate.

1. The values of b and c, on which the ſolution is founded, are derived from experiments, which are ſubject to conſiderable inaccuracies.

2. The reſiſtance of the air has a great effect in retarding the deſcent of heavy bodies, when the velocity

becomes so great as is supposed in this question; and this circumstance is not regarded in the solution.

3. A small error, in making the experiment to which this question relates, produces a great error in the conclusion. This circumstance is particularly to be attended to in all physical problems; and, in the present case, without noticing the preceding imperfections, an error of half a second, in estimating the time, makes an error of above 100 feet in the expression of the depth of the pit.

SECTION III.

OF INTEREST AND ANNUITIES.

THE application of Algebra to the calculation of Interests and Annuities, will furnish proper examples of its use in business. Algebra cannot determine the propriety or justice of the common suppositions on which these calculations are founded, but only the necessary conclusions resulting from them.

DEFINITIONS. I. *Interest* is the premium allowed for the loan of money.

II. The sum lent is called the *Principal.*

III. The sum of the principal and interest is called the *Amount.*

IV. Interest is allowed at so much *per cent. per annum,* which premium *per cent. per annum,* or interest of 100*l.* for a year, is called *the Rate of Interest.*

V. Simple

V. *Simple Interest* is that which is allowed for the principal lent only.

VI. *Compound Interest* is that which is allowed, not only for the sum lent, but also for its interest, as it becomes due at the end of each stated time of payment.

VII. *An annuity* is a payment made annually for a certain term of years.

VIII. An annuity which is to commence after a certain time, and then to continue some years, is called *an annuity in reversion*.

IX. An annuity which is to continue for ever, is termed *a perpetuity*.

X. Annuities or Pensions, &c. are said to be *in arrears* when they are payable or due, either yearly, half-yearly, or quarterly, and are unpaid for any number of payments.

Notation.

In the following theorems let p denote any principal sum of which 1*l.* is the unit, t the time during which it bears interest, of which 1 year shall be the unit, r the rate of interest of 1*l.* for 1 year, and let a be the amount of the principal sum, p, with its interest for the time, t, at the rate r.

I. *Of Simple Interest.*

THEOREM. In simple interest $a = p + prt$, and of these four, a, p, r, t, any three being given, the fourth may be found by resolving a simple equation.

The

The foundation of the canon is very obvious; for the intereſt of 1*l.* in 1 year is r, for t years it is rt, and for p pounds it is prt; the whole amount of principal and intereſt muſt therefore be $p+prt=a$.

Example I.

What will 368*l.* 16*s.* amount to in 7¾ years at 6½ per cent. per annum?

Here $p=368.8l.$ $t=7.75$ years, $r=0.065l.$ (for $100 : 6.5 :: 1 : \frac{6.5}{100}=0.065$).

$$\text{Therefore } a=368.8l.+368.8l.\times 0.065\times 7.75$$
$$=554.583l.=554l.\ 11s.\ 7d.\ 3.68f.$$

Example II.

What principal will amount to 554*l.* 11*s.* 7*d.* 3.68*f.* in 7¾ years at 6½ per cent. per annum?

Here $a=554.583l.$ $r=0.065l.$ $t=7.75$ years, conſequently

$$p=\frac{a}{1+rt}=\frac{554.583l.}{1+0.065\times 7.75}=\frac{554.583l.}{1.50375}$$
$$=368.8l.=368l.\ 16s.$$

II. *Of Compound Intereſt.*

The ſame notation being uſed, let $1+r=R$.

THEOREM. In Compound Intereſt $a=pR^t$.

For the ſimple intereſt of 1*l.* in a year is r, and the new principal ſum therefore which bears intereſt during the

the 2d year is $1+r$, or R; the interest of R for a year is rR, and the amount of principal and interest at the end of the 2d year is

$$R+rR=R\cdot\overline{1+r}=R^2,\text{ for }1+r=R.$$

In like manner, at the end of the 3rd year it is R^3, and at the end of t years it is R^t, and for the sum p it is $pR^t=a$.

COR. I. Of these four p, R, t, a, any three being given, the fourth may be found. When t is not very small, the solution will be obtained most conveniently by logarithms. When R is known, r may be found, and conversely, because $R=1+r$.

Example.

If 500*l.* has been at interest for 21 years, the whole arrear due, reckoning $4\frac{1}{2}$, or 4.5, per cent. compound interest, is 1260.12*l.* or 1260*l.* 2*s.* 4*d.* 3*f.*

In this case $p=500$, $r=.045$ (for $100 : 4.5 :: 1 : .045$), $R=1+r=1.045$, $t=21$, and $a=1260.12$, and any one of these may be derived by the theorem from the others being known.

Thus, to find a, because $R^t=(1.045)^{21}$, it will be by logarithms, $t\text{l.}R=21\text{l.}1.045=21\times0.0191163$ (by Tables) $=0.4014423$; therefore $\text{l.}R^t=0.4014423$, and $R^t=2.520242$, this being, by Tables, the number answering to the logarithm 0.4014423. Hence $a=pR^t=500\times2.520242=1260.121$.

COR. II. The present worth of a sum, s, in reversion, that is, payable after a certain time, t, is found thus:

Let

Let the present worth be x, then this money improved by compound interest, during t, produces xR^t, which must be equal to s; and therefore, if $xR^t=s$, $x=s\div R^t$.

Example.

What is the present worth of 315.6175*l.* or 315*l.* 12*s.* 4*d.* payable 4 years hence, at 6 per cent.

Here $r=0.06$, $R=1+r=1.06$, $t=4$, $s=315.6175$.

Therefore $x=\frac{315.6175}{(1.06)^4}=\frac{315.6175}{1.26247}=250l.$

Note. This operation is otherwise called *Rebate* or *Discount*.

COR. III. The time in which a sum is doubled at compound interest will be found thus:

By hypothesis $pR^t=2p$, or $R^t=2$; then, by logarithms, $t\,l.R=l.2$, and $t=l.2\div l.R$.

Example.

Suppose $r=.05$, then $R=1+r=1.05$, and

$$t=\frac{l.2}{l.1.05}=\frac{0.3010300}{0.0211893}=14.2066.$$

That is 14 years and 75 days nearly.

Note. Many other suppositions might be made with regard to the improvement of money by compound interest. The interest might be supposed to be joined to the capital, and along with it to bear interest at the end of every day, or even at the end of every instant, and suitable calculations might be formed; but these suppositions, being seldom used in practice, are omitted.

III. *Of*

III. *Of Annuities.*

The chief problem with regard to Annuities is to determine their present worth. The supposition on which the solution proceeds is, that the money received by the seller, being improved by him in a certain manner, during the continuance of the annuity, amounts to the same sum as the several payments received by the purchaser, improved in the same manner.

Let the annuity be called a, and p be the present worth of it or purchase money, t the time of its continuance, and let the other letters denote as formerly.

The seller, by improving the price received, p, at compound interest, at the time the annuity ceases, has pR^t.

The purchaser is supposed to receive the first annuity, a, at the end of the first year, which is improved by him for $t-1$ years; it becomes therefore aR^{t-1}.

He receives the second annuity at the end of the second year, and when improved, $t-2$ years, it becomes aR^{t-2}.

The third annuity becomes aR^{t-3}, &c. and the last is simple a.

Therefore the whole amount of the improved annuities is the geometrical series:

$$a+aR+aR^2+aR^3 \ldots\ldots +aR^{t-1}.$$

The sum of this series is $\frac{aR^t-a}{R-1}$ (p. .)

But,

But, from the nature of the problem, $pR^t = \frac{aR^t - a}{R - 1}$.

Therefore $R - 1 = r$, consequently $p = \frac{aR^t - a}{rR^t}$.

COR. I. If an annuity has been unpaid for the the term n, the arrear, reckoning compound interest, will be $\frac{aR^n - a}{R - 1}$, or $\frac{aR^n - a}{r}$.

COR. II. The present worth of an annuity in reversion, that is to commence after a certain time, n, and then to continue t years, is found by subtracting the present worth for n years, that is $\frac{aR^n - a}{rR^n}$, or (multiplying this fraction by R^t) $\frac{aR^{n+t} - aR_t}{rR^{n+t}}$, from the present worth for $n+t$ years, which is $\frac{aR^{n+t} - a}{rR^{n+t}}$; and then the present worth required is

$$p = \frac{aR^{n+t} - a}{rR^{n+t}} - \frac{aR^{n+t} + aR_t}{rR^{n+t}} = \frac{aR^t - a}{rR^{n+t}}.$$

COR. III. If the annuity is to continue for ever, then $t = \infty$, $R^t = R^\infty = \infty$, $R^t - 1 = \infty - 1 = \infty$, and therefore $R^t = R^t - 1$. Whence

$$p = \frac{aR^t - a}{rR^t} = a \times \frac{R^t - 1}{rR^t} = \frac{aR^t}{rR^t} = \frac{a}{r}.$$

COR. IV. A perpetuity in reverfion, fince $R^t = R^t - 1$, is $p = \frac{a}{rR^n}$, becaufe $\frac{aR^t - a}{rR^{n+t}} = \frac{aR^t}{rR^nR^t} = \frac{a}{rR^n}$.

PROBLEM. When 12 years of a leafe of 21 years were expired, a renewal for the fame term was granted for 1000*l.* 8 years are now expired, and for what fum muft a correfponding renewal be made, reckoning 5 per cent. compound intereft?

SOLUTION. From the firft tranfaction the yearly rent muft be deduced; and from this the proper fine in the fecond may be computed, as follows:

In the firft bargain, an annuity in reverfion, to commence 9 years hence, and to continue 12 years, was fold for 1000*l.* The annuity will therefore be found by Cor. II. in which all the quantities are given, but $a = \frac{prR^{n+t}}{R^{n+t} - 1}$; becaufe by inferting numbers, viz. $p = 1000$, $n = 9$, $t = 12$, $r = .05$, and $R = 1.05$; and then working by logarithms, it will be $a = 175.029 = 175l.\ 7d.$

Next, having found a, the fecond renewal is made by finding the prefent worth of the annuity a in reverfion, to commence 13 years hence, and to laft 8 years. In the canon (Cor. II.) infert for a, 175.029 and let $n = 13$, $t = 8$, $r = .05$, and $R = 1.05$ as before; and then you will find $p = 599.93 = 599l.\ 18s.\ 7d.$ the fine required.

Note. As thefe computations often become troublefome, and are of frequent ufe, all the common cafes are calculated in tables, from which the value of any annuity, for any time, at any intereft, may eafily be found.

 It

It is to be obſerved alſo, that the preceding Rules are computed on the ſuppoſition of the annuities being paid yearly; and therefore, if they be ſuppoſed to be paid half-yearly, or quarterly, the concluſions will be ſomewhat different, but they may eaſily be calculated on the foregoing principles.

The calculations of life annuities, depend partly upon the principles now explained, and partly upon phyſical principles, from the probable duration of human life, as deduced from bills of mortality.

CHAPTER VI.

OF INDETERMINATE PROBLEMS OF THE SECOND ORDER.

I.

Reſolution in Whole Numbers of the equation $y=\sqrt{(ax^2+2bx+c)}$.

PROPOSITION. To find all thoſe numbers, which ſubſtituted, inſtead of x, in the particular caſes of the general equation $x=\sqrt{(ax^2+2bx+c)}$, give all the rational and integral values of y, anſwering to thoſe particular caſes.

Caſe

Case I.

$$y=\sqrt{(2x^2+2bx+c)}.$$

Let $\sqrt{(2x^2+2bx+c)}=x+m$. Therefore
$2x^2+2bx+c=x^2+2mx+m^2$, or
$x^2+2bx-2mx=m^2-c$. Whence
$x=m-b+\sqrt{(2m^2-2bm+b^2-c}$. Now
let $\sqrt{(2m^2-2bm+b^2-c)}=m+n$. Therefore
$2m^2-2bm+b^2-c=m^2+2mn+n^2$, or
$m^2-2bm-2mn=n^2+c-b^2$. Whence
$m=n+b+\sqrt{(2n^2+2bn+c)}$, and
$x=3n-b+2\sqrt{(2n^2+2bn+c)}$. Now
this last radical expression being the same as the first given, all the business will be to find a value of x, so that the corresponding value of y be rational and integral; because such a value of x substituted, instead of n, in the said last radical expression, will give a new value of x answering to the conditions, and so on, as may be seen in the following examples.

Example II.

Let $y=\sqrt{(2x^2-1)}$.

Here we have $2b=0$, $c=-1$, and then
$x=3n+2\sqrt{(2n^2-1)}$.

Now, if $x=1$, then $y=\sqrt{(2x^2-1)}=1$. Therefore take $n=1$, and you will find $x=5$, and $y=7$,
$n=5$, - - - - - - - - - $x=29$, - - $y=41$,
$n=29$, - - - - - - - - $x=169$, - - $y=239$, &c.

Example II.

Let $y=\sqrt{(2x^2+1)}$.

Here $2b=0$, $c=1$, and $x=3n+2\sqrt{(2n^2+1)}$.

Now, if $x=0$, then $y=1$. Therefore

Take $n=0$, and it will be $x=2$, and $y=3$

$n=2$, - - - - - - $x=12$, - - $y=17$, &c.

Case II.

$$y=3x^2+2bx+c.$$

Let $\sqrt{(3x^2+2bx+c)}=x+m$. Therefore

$2x^2+2bx+c=2mx+m^2$. Whence

$$x=\frac{m-b+\sqrt{(3m^2+2bm+b^2-2c)}}{2}=m+n.$$

Hence $\sqrt{(3m^2+2bm+b^2-2c)}=m+2n+b$.

Then $m^2-2mn=2n^2+2bn+c$.

And $m=n+\sqrt{(3n^2+2bn+c)}$.

This radical expreſſion being the ſame as the firſt propoſed, it gives the reſolution required.

Example.

Let $y=\sqrt{(3xx+6)}$. Therefore

$2b=0$, $c=6$, and $x=m+n=2n+\sqrt{(3n^2+6)}$.

If $x=1$, then $y=3$. Therefore taking

$n=1$, it will be $x=5$, and $y=9$;

$n=5$, - - - - $x=19$, - - $y=33$, &c.

Note. The application of this method to the other caſes may demand a longer ſeries of operations, but, if the

the ſolution is poſſible, we ſhall always get it, only drawing from the given radical expreſſion, by the mean of convenient ſubſtitutions, another quite like it, and then finding, by trials, the leaſt value of x anſwering to our purpoſe; which value of x ſubſtituted, inſtead of n, in the general expreſſion of x, will give a new value of the ſame x, and ſo on, we ſhall have all the poſſible values required.

II.

SOLUTION OF INDETERMINATE PROBLEMS OF THE SECOND ORDER.

PROBLEM I. To divide a given ſquare number into two parts, each of of which ſhall be a ſquare number.

SOLUTION. Let the given ſquare be a^2. If one of the ſquares ſought be x^2, the other is a^2-x^2. Let $rx-a$ be a ſide of this laſt ſquare.

Then $r^2x^2-2arx+a^2=a^2-x^2$

$$r^2x^2+x^2=2arx$$

$$\div x$$

$$r^2x+x=2ar, \text{ and } x=\frac{2ar}{r^2+1}.$$

Hence $rx-a=\dfrac{2ar^2}{r^2+1}-a=\dfrac{ar^2-a}{r^2+1}.$

Let r therefore be aſſumed at pleaſure, and $\dfrac{2ar}{r^2+1}$,

$\dfrac{ar^2-a}{r^2+1}$,

$\frac{ar^2-a}{r^2+1}$, which muſt always be rational, will be the ſides of the two ſquares required.

Thus, if $a^2=100$. Then, if $r=2$, the ſides of the two ſquares are 8 and 6; therefore $64+36=100$. Alſo let $a^2=64$; then if $r=2$, the ſides of the two ſquares are $\frac{32}{5}$ and $\frac{24}{5}$; therefore $\frac{1024}{25}+\frac{576}{25}=\frac{1600}{25}=64$.

PROBL. II. To find two ſquare numbers whoſe difference is given.

SOLUTION. Let x^2 and y^2 be the ſquare numbers, and a their difference.

Put $\frac{m+n}{2}=x$, and $\frac{m-n}{2}=y$.

Then by taking the ſquares, $\frac{m^2+2mn+n^2}{4}=x^2$

$$\frac{m^2-2mn+n^2}{4}=y^2$$

$$* \quad mn \quad * = x^2-y^2=a.$$

If x and y are required only to be rational, then take n at pleaſure, and $m=a\div n$, whence x and y are known and rational.

But, if x and y are required to be whole numbers, take for m and n any two factors that produce a, and are both even or both odd numbers. And this is poſſible only where a is either an odd number greater than 1, or a number diviſible by 4. Then $\frac{m+n}{2}$ and $\frac{m-n}{2}$ are the numbers ſought. For the product of two odd numbers is odd, and that of two even numbers is diviſible by 4. Alſo, if m and n are both odd or both even, $\frac{m+n}{2}$ and $\frac{m-n}{2}$ muſt be integers.

Example

Example I.

If $a=27$, take $n=1$, then $m=27$; and the ſquares are 196 and 169: or n may be 3 and $m=9$, and then the ſquares are 36 and 9.

Example II.

If $a=12$, take $n=2$, and $m=6$; and the ſquares are 16 and 4.

PROBL. III. To find two ſquare numbers, whoſe ſum ſhall alſo be a ſquare number.

SOLUTION. Let m^2-n^2 be the ſide of one ſquare number, and $2mn$ the ſide of the other.

Then the two ſquare numbers will be $m^4-2m^2n^2+n^4$ and $4m^2n^2$, whoſe ſum $m^4+m^2n^2+n^4$ is alſo a ſquare number, the ſide of which is m^2+n^2.

Take m and n at pleaſure; then m^2-n^2 and $2mn$ will be the ſides of the two ſquare numbers ſought.

Example.

Let $m=2$, $n=1$; then $m^2-n^2=3$, $2mn=4$, and the ſquares are 16 and 9, whoſe ſum is the ſquare number 25.

PROBL. IV. To find two ſquare numbers, whoſe difference ſhall alſo be a ſquare number.

SOLUTION. Let m^2+n^2 be the ſide of one ſquare number, and $2mn$ the ſide of the other.

Then the two ſquare numbers will be $m^4+2m^2n^2+n^4$ and $4m^2n^2$, whoſe difference $m^4-2m^2n^2+n^4$ is alſo a ſquare number, the ſide of which is m^2-n^2.

Take

Take m and n at pleaſure; then m^2+n^2 and $2mn$ will be the ſides of the two ſquare numbers ſought.

Example.

Let $m=2$, $n=1$; then $m^2+n^2=5$, $2mn=4$, and the ſquares are 25 and 16, whoſe difference 9 is alſo a ſquare number.

PROBL. V. To find all thoſe whole numbers, whoſe ſquares, being doubled, ſurpaſs unity by another ſquare.

SOLUTION. Let x be one of theſe numbers; then $2xx-1$, by the conditions, muſt be a ſquare, that is, $y^2=2xx-1$, which is the firſt Example conſidered in Caſe I. of the Propoſition.

PROBL. VI. To find all thoſe whole numbers, whoſe ſquares being doubled and added to unity produce other ſquares.

SOLUTION. Let x be one of theſe numbers; then $2xx+1$, by the conditions muſt be a ſquare, that is, $y^2=2xx+1$, which is Example II. of Caſe I. of the Propoſition.

PROBL. VII. To find all thoſe whole numbers, whoſe ſquares being tripled and added to 6, produce other ſquares.

SOLUTION. Let x be one of theſe numbers; then $3x^2+6=y^2$, which is the Example of Caſe II. of the Propoſition.

CHAP-

CHAPTER VII.

A PROMISCUOUS COLLECTION OF PROBLEMS.

I. *Determinate Problems of the 1st Order.*

1. Let m and n be two given multipliers. It is required to divide a given number, as a, into two ſuch parts, called x and y, that mx added ny may make ſome other given number, as b.

Anſ. $x = \frac{b - an}{m - n}$ and $y = \frac{am - b}{m - n}$.

2. To divide a given number a, into two ſuch parts, x and y, that x may be to y as m to n.

Anſ. $x = \frac{am}{m+n}$ and $y = \frac{an}{m+n}$.

3. To find a number x, which being ſeverally added to two given numbers, a and b, will make the former ſum to the latter as m to n.

Anſ. $x = \frac{an - bm}{m - n}$.

4. To divide a given number, a, into two ſuch parts, x and y, that the exceſs of x above another given number, as b, may be to what y wants of b, as m to n; ſuppoſing m to be greater than n.

Anſ. $x = \frac{am - bm - bn}{m - n}$ and $y = \frac{bm + bn - an}{m - n}$.

5. A gentleman meeting a company of beggars, gives to each n pence, and has a pence over; but if he had given them m pence apiece, he would have found he had wanted b pence for that purpoſe: I demand the numbers of beggars, x, and the pence which the gentleman had, y.

Anſ. $x = \frac{a + b}{m - n}$ and $y = \frac{am + bn}{m - n}$.

6. There are two places whoſe diſtance from each other is a, and from whence two perſons, A and B, ſet out at the ſame time with a deſign to meet, A travelling at the rate of p miles in q hours, and B at the rate of r miles in s hours: I demand how long and how far each travelled before they met.

Anſ. The number of hours travelled by each, $\frac{aqs}{ps + qr}$.

Miles travelled by A, $\frac{aps}{ps + qr}$.

Miles travelled by B, $\frac{aqr}{ps + qr}$.

7. Out of a common pack of fifty-two cards, let part be diſtributed into ſeveral diſtinct parcels or heaps in the following manner: upon the loweſt card of every heap let as many others be laid as are ſufficient to make up its number twelve; as if four be the number of the loweſt card, let eight others be laid upon it; if five, let ſeven; if a let $12 - a$, &c. It is required, having given the number of heaps, which we ſhall call n, as alſo the number of cards ſtill remaining in the dealer's hand, which we ſhall call r, to find the ſum, x, of the numbers

numbers of all the bottom cards put together.

Anf. $x = 13n - 52 + r = 13 \times \overline{n-4} + r.$

8. Let n be the number of heaps as before, p the number of cards in a pack; let as many cards be laid upon the lowest of every heap as are sufficient to make up its number q; and lastly, let r be the number of remaining cards as before: it is required, to find the sum x of the numbers of all the bottom cards put together.

Anf. $x = nq + n + r - p \pm \overline{q+1} \times n + r - p.$

9. Three causes C, C′, C″, working separately, may produce the effects E, E′, E″, in the times T, T′, T″. In what time will they, working jointly, produce the effect E‴?

Anf. $T''' = \dfrac{E'''}{E \div T + E' \div T' + E'' \div T''}.$

10. The particular rates a and b of two ingredients to be mixed, and the rate c of the mixture m being given, to find what portions x and y of each ingredient must be taken to compose that mixture.

Anf. $x = m \times \dfrac{c-b}{a-b}$ and $y = m \times \dfrac{a-c}{a-b}.$

11. Two travellers, A and B, whose velocities are to one another, as m to n, set out from two places C and D at the same time, A from C bound for D, and B from D bound for C; it is required to find, at what distance, x, from the place C they will meet together.

Anf. $x = \dfrac{md}{m+n}.$

12. A hare having made a number n of leaps, a greyhound begins to pursue her. The greyhound takes

m leaps in the ſame time that the hare takes $m+a$; but the leaps of the greyhound are greater than thoſe of the hare in the proportion of p to q. It is required to find a canon, in order to know, whether the greyhound ſhall overtake the hare, and at what diſtance, x, calculated in leaps of the greyhound.

Anſ. $x=\frac{mnp}{qm-pa-pm}$, and the greyhound will overtake the hare only when $qm>pa+pm$.

II. *Determinate Problems of the ſecond and higher Orders.*

1. A certain company at a tavern had a reckoning of a pounds to pay; upon which a number, b, of the company going away, obliged the reſt to pay c ſhillings apiece more than they ſhould have done: what was the number, x, of perſons?

Anſ. $x=\frac{b}{2}\pm\sqrt{\frac{ab}{c}+\frac{bb}{4}}$.

2. To divide the number a into two ſuch parts, x being the leſs, and y the greater, that my multiplied by nx may give the product b.

Anſ. $x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{b}{mn}}$ and $y=a-x$.

3. Two merchants, A and B, enter into partnerſhip with a principal of a pounds together; the merchant A puts in his money, x, for m months; and the other B his money, y, for n months; then each of them receives b pounds, principal and intereſt. What are their ſeveral principals?

Anſ. $x=-p\pm\sqrt{q+p^2}$ and $y=a-x$.

Note.

Note. Here we suppose $p=\frac{an+bm+bn-am}{m-n}$, and $q=\frac{abn}{m-n}$.

4. The sum, a, of two numbers, x and y, and the quotient, q, which arises from the division of the less, x, by the cubic root of the greater, y, being given, to find these numbers.

$$\text{Anf.}\quad x=q\sqrt[3]{\frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{q^3}{27}}}+q\sqrt[3]{\frac{a}{2}-\sqrt{\frac{a^2}{4}+\frac{q^3}{27}}}$$

and $y=a-x$.

5. A certain number, x, of persons enter into partnership, and each of them contributes m times as many pounds, as there are persons: they gained for every 100 pounds so many pounds as there are persons and n more, that is, $x+n$ pounds; lastly the total gain is a pounds. I demand the number of persons.

Anf. $mx^3+mnx^2-100a=0$.

6. Some merchants have in common a principal of a pounds; each of them adds to it m times as many pounds as there are partners; they gain as many pounds per cent. as there are partners; and after having taken each n times as many pounds, as there are partners, there remain b pounds. What is the number, x, of merchants?

Anf. $mx^3-100nx^2+ax-100b=0$.